# Polygonal Heat Conductors with a Stationary Hot Spot *

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**Abstract**

We consider a convex polygonal heat conductor whose inscribed circle touches every side of the conductor. Initially, the conductor has constant temperature and, at every time, the temperature of its boundary is kept at zero. The hot spot is the point at which temperature attains its maximum at each given time. It is proved that, if the hot spot is stationary, then the conductor must satisfy two geometric conditions. In particular, we prove that these geometric conditions yield some symmetries provided the conductor is either pentagonal or hexagonal.

**Key words.** heat equation, hot spot, polygonal conductor, initial behavior, symmetries of domains.

**AMS subject classifications.** Primary 35K05, 35B38, ; Secondary 35B40, 35K20.

## 1 Introduction

A hot spot in a heat conductor is a point at which temperature attains its maximum at each given time. Let $\Omega$ be a bounded convex domain in the Euclidean space $\mathbb{R}^N,\ N \geq 2$, and consider a heat conductor $\Omega$ having initial constant temperature and zero boundary temperature at every time. The physical situation can be modeled as the following initial-boundary value problem for the heat equation:

$$u_t = \Delta u \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{1.2}$$

$$u = 1 \quad \text{on} \quad \Omega \times \{0\}, \tag{1.3}$$

where $u = u(x, t)$ denotes the normalized temperature at a point $x \in \Omega$ and at a time $t > 0$.

Since $\Omega$ is convex, a result of [BL] shows that $\log u(x, t)$ is concave in $x$, which, together with the analyticity of $u$ in the spatial variable $x$, implies that for each time $t > 0$ there exists a unique point $x(t) \in \Omega$ satisfying

$$\{x \in \Omega \ : \ \nabla u(x, t) = 0\} = \{x(t)\}, \tag{1.4}$$

where $\nabla$ denotes the spatial gradient. The point $x(t)$ is the unique hot spot for each time $t > 0$. Put $\mathcal{M} = \{x \in \Omega \ : \ d(x) = \max_{z\in\Omega} d(z)\}$, where $d(z)$ is the distance of $z$ to $\partial\Omega$ defined by

$$d(z) = \text{ dist}(z, \partial\Omega) \ (= \inf\{|z - y| \ : \ y \in \partial\Omega\}) \quad \text{for } z \in \overline{\Omega}. \tag{1.5}$$

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Then we have

$$\operatorname{dist}(x(t), \mathcal{M}) \to 0 \quad \text{as } t \to 0^+, \tag{1.6}$$

since the function $-4t\log[1-u(x,t)]$ attains its maximum at $x = x(t)$ for each $t > 0$ and a result of Varadhan [V] shows that

$$-4t\log[1-u(x,t)] \to d(x)^2 \quad \text{as } t \to 0^+ \text{ uniformly on } \overline{\Omega}. \tag{1.7}$$

In conclusion, the hot spot $x(t)$ starts from $\mathcal{M}$. Also, as $t \to \infty$, $x(t)$ tends to the point at which the positive first eigenfunction of $-\Delta$ with the homogeneous Dirichlet boundary condition attains its maximum (see [MS 3], Introduction, for details).

From now on, without loss of generality, we shall assume that $\Omega$ contains the origin 0.

A conjecture of Klamkin [Kl] stated that, if the origin is a stationary hot spot, that is, if $x(t) \equiv 0$, then $\Omega$ must be centro-symmetric with respect to 0. This was denied by Gulliver-Willms [GW] and Kawohl [Ka]. A typical counterexample is an equilateral triangle in the plane. After that Chamberland-Siegel [CS] posed the following conjecture.

**Conjecture 1.1** (Chamberland–Siegel) *If 0 is a stationary hot spot in a bounded convex domain $\Omega$, then $\Omega$ is invariant under the action of an essential subgroup $G$ of orthogonal transformations.*

A subgroup $G$ of orthogonal transformations is said to be *essential* if, for every $x \neq 0$, there exists an element $g \in G$ such that $gx \neq x$. As observed in [CS], it is quite easy to prove that, if $\Omega$ is invariant under the action of an essential subgroup $G$ of orthogonal transformations, then the origin must be a stationary hot spot. Indeed, if $\Omega$ enjoys that invariance, then by the unique solvability of the initial-Dirichlet problem (1.1)-(1.3) the solution $u$ itself is invariant under the action of $G$. Namely, we have $u(x,t) \equiv u(gx,t) \quad (x \in \Omega,\ t > 0,\ g \in G)$. Taking the gradient of both sides of the last identity, together with the assumption that $G$ is essential, implies that $\nabla u(0,t) = 0$ $(t > 0)$, and then it follows from (1.4) that the origin is a stationary hot spot.

A proof of Conjecture 1.1 appears to be a much harder task. So far, the only known result in this direction is the following theorem, that was proved by the authors in [MS 3] as a consequence of a more general one.

**Theorem 1.2** *Let $\Omega$ be a bounded domain in $\mathbb{R}^2$. Then the following hold true.*

(1) *If $\Omega$ is a triangle and 0 is a stationary hot spot, then $\Omega$ must be an equilateral triangle centered at 0.*

(2) *If $\Omega$ is a convex quadrangle and 0 is a stationary hot spot, then $\Omega$ must be a parallelogram centered at 0.*

(3) *If $\Omega$ is a non-convex quadrangle, then there is no stationary critical point of $u$ in $\Omega$. In particular, there is no stationary hot spot.*

In (1) of Theorem 1.2, $G$ is the cyclic group generated by the rotation of the angle $\frac{2\pi}{3}$, and in (2) $G = \{I, -I\}$ where $I$ is the identity mapping. The proof is based on two ingredients; one is the balance law around stationary critical points of the heat flow (see [MS 1]) and the other makes use of the asymptotic behavior as $t \to 0^+$ of solutions of the heat equation due to Varadhan [V].

In the present paper, we are able to treat the case of certain pentagons and hexagons, as the following result specifies.

**Theorem 1.3** *Let $\Omega$ be a convex polygon in $\mathbb{R}^2$ and suppose that its inscribed circle touches every side of $\Omega$. Then the following propositions hold true.*

(1) *If* $\Omega$ *is a pentagon and* $0$ *is a stationary hot spot, then* $\Omega$ *must be a regular pentagon centered at* $0$.

(2) *If* $\Omega$ *is a hexagon and* $0$ *is a stationary hot spot, then* $\Omega$ *is invariant under the action of the rotation of one of angles* $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\pi$.

This theorem is a consequence of the following general statement.

**Theorem 1.4** *Let* $\Omega$ *be a convex polygon in* $\mathbb{R}^2$ *with* $m$ *sides,* $m \geq 5$, *and let* $B_R(0)$ *be an open disk with radius* $R > 0$ *and centered at* $0$.

*Suppose that* $0$ *is a stationary hot spot and the circle* $\partial B_R(0)$ *touches every side of* $\Omega$ *at the points* $p_1, \cdots, p_m \in \partial\Omega \cap \partial B_R(0)$. *Let* $q_1, \cdots, q_k$ *be the* $k$ $(1 \leq k \leq m)$ *nearest vertices of* $\Omega$ *to* $0$.

*Then we have that*

$$\sum_{i=1}^{m} p_i = 0 \tag{1.8}$$

*and*

$$\sum_{j=1}^{k} q_j = 0. \tag{1.9}$$

We observe that, in the special case in which the vertices $q_1, \ldots, q_k$ are consecutive, equation (1.9) easily implies that $k = m$ and $\Omega$ must be a regular polygon.

Notice that (1.8) was already obtained in [MS 3]. However, (1.9) is new and is derived by coupling a suitable extension argument to a careful analysis of the short-time behavior of $u(x,t)$ near the vertices of $\Omega$.

The present paper is organized as follows. Both Section 2 and Section 3 are devoted to the proof of Theorem 1.4. In Section 2, we introduce the function $v = 1 - u$ and give sub- and supersolutions $v^-$, $v^+$ for the initial-boundary value problem solved by $v$. Then, by folding back $v$ with respect to each side of $\Omega$, we extend $v$ to a solution of the heat equation in a domain larger than $\Omega$ and by using the balance law around a stationary critical point, we obtain (1.8) and the main identity (2.13). In Section 3, with the aid of $v^-$, $v^+$, we exploit a more detailed initial behavior of $v$ and eventually obtain (1.9). Finally, in Section 4, by using Theorem 1.4, we prove Theorem 1.3.

# 2 Barriers for an extension of the solution

In this section, we shall extend the solution of (1.1)-(1.3) to a larger domain, in order to prove (1.8) and prepare the proof of (1.9).

Let $\Omega$ be a convex $m$-gon in $\mathbb{R}^2$ with $m \geq 5$. Suppose that the circle $\partial B_R(0)$ touches every side of $\Omega$, say $\partial\Omega \cap \partial B_R(0) = \{p_1, \cdots, p_m\}$. Let $q_1, \cdots, q_k$ be the $k$ $(1 \leq k \leq m)$ nearest vertices of $\Omega$ to the origin; we can set $R^* = |q_1| = |q_2| = \cdots = |q_k|$, and hence $R^* > R$.

Denote by $\nu_1, \cdots .\nu_m$ the interior normal unit vectors to $\partial\Omega$ at the points $p_1, \cdots, p_m$, respectively. Note that

$$p_i = -R\nu_i \quad (i = 1, \cdots, m). \tag{2.1}$$

For notational convenience, we deal with the function $v = 1 - u$ instead of $u$ and consider the cold spot of $v$ instead of the hot spot of $u$; then $v$ satisfies:

$$v_t = \Delta v \quad \text{in} \quad \Omega \times (0, \infty), \tag{2.2}$$

$$v = 1 \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{2.3}$$

$$v = 0 \quad \text{on} \quad \Omega \times \{0\}. \tag{2.4}$$

We now introduce a subsolution $v^- = v^-(x,t)$ and a supersolution $v^+ = v^+(x,t)$ for problem (2.2)-(2.4).

Let $f = f(\xi)$ be the function defined by

$$f(\xi) = \frac{1}{\sqrt{\pi}} \int_\xi^\infty e^{-\frac{1}{4}\eta^2} d\eta \quad \text{for all } \xi \in \mathbb{R}; \tag{2.5}$$

note that

$$\int_0^\infty \xi f(\xi)\ d\xi = 1. \tag{2.6}$$

The function $w = w(s,t)$ given by

$$w(s,t) = f\left(t^{-\frac{1}{2}} s\right) \text{ for } (s,t) \in \mathbb{R} \times (0,\infty) \tag{2.7}$$

satisfies the one-dimensional heat equation $w_t = w_{ss}$ in $\mathbb{R} \times (0,\infty)$. Hence, we easily see that the functions defined by

$$v^-(x,t) = \max_{1 \le i \le m} f\left(t^{-\frac{1}{2}}(x - p_i) \cdot \nu_i\right), \tag{2.8}$$

$$v^+(x,t) = \sum_{i=1}^m f\left(t^{-\frac{1}{2}}(x - p_i) \cdot \nu_i\right). \tag{2.9}$$

are respectively a sub- and a supersolution for problem (2.2)-(2.4). By the comparison principle it follows that

$$v^- \le v \le v^+ \text{ in } \Omega \times (0,\infty). \tag{2.10}$$

The following lemma will be useful in Section 3.

**Lemma 2.1** *For any compact set $K$ contained in $\Omega$, there exist two positive constants $A > 0, B > 0$ satisfying*

$$0 < v(x,t) \le Ae^{-\frac{B}{t}} \quad \textit{for all } (x,t) \in K \times (0,\infty).$$

*Proof.* This lemma follows directly from (2.10) and from the convexity of $\Omega$. $\Box$

Note that Lemma 2.1 holds true also for general domain (not necessarily convex) $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ because of Varadhan's result (1.7).

By following the procedure employed in [MS 3], we extend $v$ to a solution $v^* = v^*(x,t)$ of the heat equation in a larger domain domain $\Omega^* \times (0,\infty) \supset \Omega \times (0,\infty)$. $\Omega^*$ is obtained by putting together $\Omega$ and all its reflections with respect to each of its sides and by eliminating possible overlaps; $v^*$ equals $1 - u^*$, where $u^*$ is obtained by odd reflections of $u$ with respect to each side of $\Omega$. It is clear that $B_{R^*}(0) \subset \Omega^*$ (see Fig. 1 (a)).

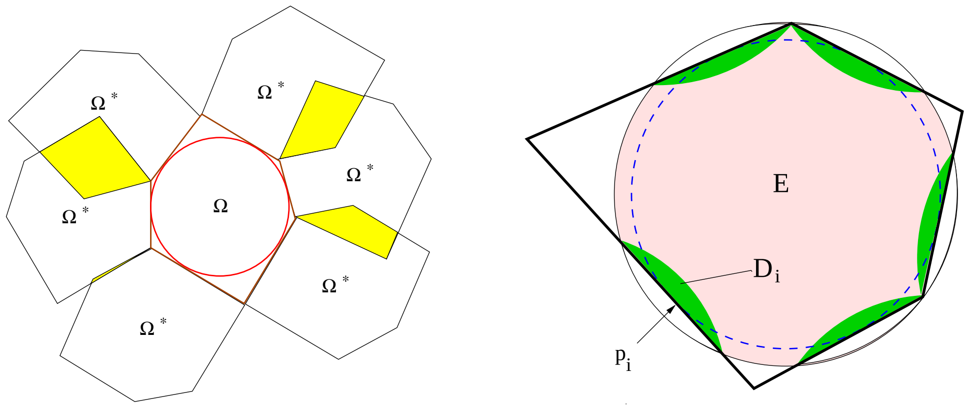
Since 0 is a stationary cold spot of $v$, we infer that it is a stationary critical point of $v^*$.

Therefore we can use the balance law obtained in [MS 1], Theorem 2 (see also [MS 2], Corollary 2.2, for another proof) to infer that

$$\int_{B_{R^*}(0)} x v^*(x,t)\ dx = 0 \text{ for any } t > 0. \tag{2.11}$$

Letting $t \to 0^+$ yields that

$$2 \int_{B_{R^*}(0) \setminus \Omega} x\ dx = 0, \tag{2.12}$$




(a) (b)

Figure 1: (a) The construction of the set $\Omega^*$ and (b) the sets $E$, $D_i$ and $D = \bigcup D_i$.

since $v^*$ tends to 0 inside $\Omega$ and to 2 outside; (2.12) easily implies (1.8).

Denote by $D$ the region obtained as the union of the reflections of each connected component of $B_{R^*}(0) \setminus \overline{\Omega}$ with respect to each relevant side of $\Omega$, let $D_j, 1 \le j \le m$, be the connected components of $D$, and put $E = (B_{R^*}(0) \cap \Omega) \setminus \overline{D}$. Note that both $D$ and $E$ are contained in $\Omega$. For $x \in D_j, 1 \le j \le m$, denote by $x^*$ the reflection of $x$ with respect to the side of $\Omega$ containing $\overline{D}_j \cap \partial\Omega$. Then $v^*(x,t) \equiv 2 - v^*(x^*,t)$ because of (2.3) (see Fig. 1 (b)). Since

$$\int_{B_{R^*}(0)} x v^*(x,t)\, dx =$$
$$\int_E x v(x,t)\, dx + \int_D x v(x,t)\, dx + \int_{B_{R^*}(0)\setminus(D\cup E)} x v(x,t)\, dx =$$
$$\int_E x v(x,t)\, dx + \int_D x v(x,t)\, dx + \int_D x^*[2 - v(x,t)]\, dx,$$

from (2.11) and (2.12) it follows that for any $t > 0$

$$\int_E x v(x,t)\, dx + \int_D (x - x^*)\, v(x,t)\, dx = 0. \tag{2.13}$$

In order to prove (1.9), with the help of (2.10), in the next section we shall compute the limit

$$\lim_{t\to 0^+} \frac{1}{t} \left\{ \int_E x v(x,t)\, dx + \int_D (x - x^*)\, v(x,t)\, dx \right\}. \tag{2.14}$$

# 3 Proof of Theorem 1.4: asymptotic lemmas

When $k < m$, let $s_1, \cdots, s_\ell \quad (\ell = 2m - 2k)$ be all the points such that

$$\partial\Omega \cap \partial B_{R^*}(0) = \{q_1, \cdots, q_k, s_1, \cdots, s_\ell\}. \tag{3.1}$$

Since each $p_i$ is the midpoint of a pair of points in $\partial\Omega \cap \partial B_{R^*}(0)$, from (1.8) we have:

$$2\sum_{j=1}^{k} q_j + \sum_{j=1}^{\ell} s_j = 2\sum_{i=1}^{m} p_i = 0. \tag{3.2}$$

Notice that, when $k = m$, the definition of the points $q_1, \cdots, q_m$ implies that all the angles of $m-$gon $\Omega$ must be equal to each other, and hence $\Omega$ must be a regular polygon. Thus (1.9) holds true when $k = m$. Hereafter we assume that $k < m$.

Since the circle $\partial B_R(0)$ touches every side of $\Omega$, all the angles between the circle $\partial B_{R^*}(0)$ and the sides of $\Omega$ at $q_j$ or at $s_j$ are equal. Denote by $\alpha \in (0, \frac{\pi}{2})$ these angles.

In view of Lemma 2.1, it is enough to replace the sets in the integrals in (2.14) with small neighborhoods of the points $q_j, s_j$, and small neighborhoods of $\partial\Omega$ in $D_j$. Choose a number $\delta_0 > 0$ so small that, for any $x \in \{q_1, \cdots, q_k, s_1, \cdots, s_\ell\}$,

$$\overline{B_{\delta_0}(x)} \cap (\{p_1, \cdots, p_m, s_1, \cdots, s_\ell\} \cup \{ \text{ vertices of } \Omega\}) = \{x\}.$$

**Lemma 3.1** *For $\varepsilon > 0$ and $1 \leq j \leq \ell$ set*

$$E^\varepsilon(s_j) = \{x \in E : 0 < (x - s_j) \cdot \nu_i < \varepsilon\} \cap B_{\delta_0}(s_j),$$

*where $\nu_i$ is the interior unit normal vector to the side of $\Omega$ containing the point $s_j$ (see Fig. 2).*

*Then, if $\varepsilon$ is sufficiently small, we have:*

$$\lim_{t\to 0^+} \frac{1}{t} \int_{E^\varepsilon(s_j)} x\ v(x,t)\ dx = 2 \cot\alpha\ s_j \quad \text{for } 1 \leq j \leq \ell. \tag{3.3}$$

*Proof.* Since $\Omega$ is convex and $s_j$ is not a vertex of $\Omega$, (2.8), (2.9), and (2.10) imply that there exist two positive constants $A_j$ and $B_j$ such that

$$\left| v(x,t) - f\left(t^{-\frac{1}{2}}(x - s_j) \cdot \nu_i\right) \right| \leq A_j e^{-\frac{B_j}{t}} \ \text{ for all } \ x \in \Omega \cap B_{\delta_0}(s_j), t > 0. \tag{3.4}$$

Here we have used the fact that $(x - s_j) \cdot \nu_i = (x - p_i) \cdot \nu_i$.

Set $e_i = (p_i - s_j)/|p_i - s_j|$; if $\varepsilon > 0$ is sufficiently small, we can write

$$E^\varepsilon(s_j) = \{x = s_j + z_1\ \nu_i + z_2\ e_i : 0 < z_1 < \varepsilon,\ \varphi_-(z_1) < z_2 < \varphi_+(z_1)\},$$

where $\varphi_-(z_1) < 0 < \varphi_+(z_1)$ for $z_1 \in (0, \varepsilon)$ and the functions $\varphi_-$ and $\varphi_+$ represent respectively $\partial E^\varepsilon(s_j) \cap \partial B_{R^*}(0)$ and $\partial E^\varepsilon(s_j) \cap \partial D$. Note that $\varphi_-'(0) = -\cot\alpha$ and $\varphi_+'(0) = \cot\alpha$.

In view of (3.4), we calculate

$$\frac{1}{t} \int_{E^\varepsilon(s_j)} f\left(t^{-\frac{1}{2}}(x - s_j) \cdot \nu_i\right)\ dx =$$

$$\frac{1}{t} \int_0^\varepsilon \left[ f\left(t^{-\frac{1}{2}} z_1\right) \int_{\varphi_-(z_1)}^{\varphi_+(z_1)} dz_2 \right]\ dz_1 = \int_0^{t^{-\frac{1}{2}}\varepsilon} \frac{\varphi_+(t^{\frac{1}{2}}\xi) - \varphi_-(t^{\frac{1}{2}}\xi)}{t^{\frac{1}{2}}\xi}\ \xi f(\xi) d\xi.$$

Since

$$\lim_{t\to 0^+} \frac{\varphi_+(t^{\frac{1}{2}}\xi) - \varphi_-(t^{\frac{1}{2}}\xi)}{t^{\frac{1}{2}}\xi} = \varphi_+'(0) - \varphi_+'(0) = 2\cot\alpha \qquad \text{for } \xi > 0,$$

by Lebesgue's dominated convergence theorem we get

$$\lim_{t\to 0^+} \frac{1}{t} \int_{E^\varepsilon(s_j)} f\left(t^{-\frac{1}{2}}(x - s_j) \cdot \nu_i\right)\ dx = 2\cot\alpha \int_0^\infty \xi f(\xi)\ d\xi. \tag{3.5}$$

In a similar way, we obtain

$$\lim_{t\to 0^+} \frac{1}{t} \int_{E^\varepsilon(s_j)} (x - s_j)\ f\left(t^{-\frac{1}{2}}(x - s_j) \cdot \nu_i\right)\ dx = 0, \tag{3.6}$$

since

$$\lim_{t\to 0^+}\frac{1}{t}\int_0^{\varepsilon}\left[f\left(t^{-\frac{1}{2}}z_1\right)\int_{\varphi_-(z_1)}^{\varphi_+(z_1)} z_i\ dz_2\right]\ dz_1=0\ \ \text{for}\ \ i=1,2.$$

With the aid of (3.4), (3.5), (3.6), and (2.6) we then get (3.3). □

**Lemma 3.2** *For $\varepsilon>0$ and $1\le j\le k$ set*

$$E^{\varepsilon}(q_j)=\{x\in E: 0<(x-q_j)\cdot\nu_i<\varepsilon\ \textit{or}\ 0<(x-q_j)\cdot\nu_{i+1}<\varepsilon\}\cap B_{\delta_0}(q_j),$$

*where $\nu_i$ and $\nu_{i+1}$ are the interior unit normal vectors to the two sides of $\Omega$ containing the vertex $q_j$ (see Fig. 2).*

*Then, if $\varepsilon$ is sufficiently small, we have that*

$$4\cot 2\alpha\le\limsup_{t\to 0^+}\frac{1}{t}\int_{E^{\varepsilon}(q_j)} v(x,t)\ dx\le 8\cot 2\alpha, \tag{3.7}$$

*and*

$$\lim_{t\to 0^+}\frac{1}{t}\int_{E^{\varepsilon}(q_j)}(x-q_j)\ v(x,t)\ dx=0, \tag{3.8}$$

*for $1\le j\le k$.*

*Proof.* Let $\beta$ be the angle of $\Omega$ at the vertex $q_j$; observe that $\beta+2\,\alpha=\pi$. Since $\beta$ is the largest angle in $\Omega$, we have that $\pi(1-2/m)<\beta<\pi$, $\alpha<\pi/m$ and hence

$$\beta-2\,\alpha>0,$$

for every $m\ge 4$.

Let $\gamma$ be the bisectrix of the angle of $\Omega$ at $q_j$; $\gamma$ divides $E^{\varepsilon}(q_j)$ into two parts, $E_i^{\varepsilon}(q_j)$ and $E_{i+1}^{\varepsilon}(q_j)$, corresponding to $\nu_i$ and $\nu_{i+1}$, respectively.

Since $q_j$ is a vertex of $\Omega$, (2.8), (2.9), and (2.10) imply that there exist two positive constants $A_j$ and $B_j$ such that

$$\begin{aligned}&0<f\left(t^{-\frac{1}{2}}(x-q_j)\cdot\nu_i\right)\le v(x,t)\le 2\ f\left(t^{-\frac{1}{2}}(x-q_j)\cdot\nu_i\right)+A_j e^{-\frac{B_j}{t}}\\&\qquad\text{for all}\ \ x\in E_i^{\varepsilon}(q_j),\ t>0.\end{aligned} \tag{3.9}$$

Here we have used the fact that $(x-q_j)\cdot\nu_i=(x-p_i)\cdot\nu_i$.

Set $e_i=(p_i-q_j)/|p_i-q_j|$; if $\varepsilon$ is sufficiently small, we can write

$$E_i^{\varepsilon}(q_j)=\{x=q_j+z_1\ \nu_i+z_2\ e_i: 0<z_1<\varepsilon,\ z_1\tan\alpha<z_2<\varphi(z_1)\}.$$

Note that $\varphi'(0)=\cot\alpha$ and $\varphi'(z_1)>0$ for $z_1>0$.

We now write:

$$\frac{1}{t}\int_{E_i^{\varepsilon}(q_j)} f\left(t^{-\frac{1}{2}}(x-q_j)\cdot\nu_i\right)\ dx=\frac{1}{t}\int_0^{\varepsilon}\left[f\left(t^{-\frac{1}{2}}z_1\right)\int_{z_1\tan\alpha}^{\varphi(z_1)}dz_2\right]\ dz_1=$$

$$\frac{1}{t}\int_0^{\varepsilon}f\left(t^{-\frac{1}{2}}z_1\right)[\varphi(z_1)-z_1\tan\alpha]\ dz_1=\int_0^{t^{-\frac{1}{2}}\varepsilon}\xi f(\xi)\ \frac{\varphi(t^{\frac{1}{2}}\xi)-t^{\frac{1}{2}}\xi\tan\alpha}{t^{\frac{1}{2}}\xi}\ d\xi.$$

Thus, since

$$\lim_{t\to 0^+}\frac{\varphi(t^{\frac{1}{2}}\xi)}{t^{\frac{1}{2}}\xi}=\varphi'(0)=\cot\alpha\quad\ \text{for}\ \ \xi>0,$$

by Lebesgue's dominated convergence theorem we get

$$\lim_{t\to 0^+}\frac{1}{t}\int_{E_i^\varepsilon(q_j)} f\left(t^{-\frac{1}{2}}(x-q_j)\cdot\nu_i\right)\,dx = 2\cot 2\alpha\int_0^\infty \xi f(\xi)\,d\xi. \tag{3.10}$$

By a similar calculation, we have

$$\lim_{t\to 0^+}\frac{1}{t}\int_{E_i^\varepsilon(q_j)} |x-q_j|\,f\left(t^{-\frac{1}{2}}(x-q_j)\cdot\nu_i\right)\,dx = 0, \tag{3.11}$$

since

$$\lim_{t\to 0^+}\frac{1}{t}\int_0^\varepsilon\left[f\left(t^{-\frac{1}{2}}z_1\right)\int_{z_1\tan\alpha}^{\varphi(z_1)} z_i\,dz_2\right]dz_1 = 0\ \text{ for }\ i=1,2.$$

From (3.9), (3.10), and (2.6) it follows that

$$2\cot 2\alpha \le \limsup_{t\to 0^+}\frac{1}{t}\int_{E_i^\varepsilon(q_j)} v(x,t)\,dx \le 4\cot 2\alpha. \tag{3.12}$$

Also, since

$$\left|\frac{1}{t}\int_{E_i^\varepsilon(q_j)}(x-q_j)\,v(x,t)\,dx\right| \le \frac{1}{t}\int_{E_i^\varepsilon(q_j)}|x-q_j|\,v(x,t)\,dx,$$

we have from (3.9) and (3.11)

$$\lim_{t\to 0^+}\frac{1}{t}\int_{E_i^\varepsilon(q_j)}(x-q_j)\,v(x,t)\,dx = 0.$$

By the same arguments we obtain the last two formulas with $E_i^\varepsilon(q_j)$ replaced by $E_{i+1}^\varepsilon(q_j)$, and hence (3.7) and (3.8) follow at once. □

**Lemma 3.3** *For any* $j,s\in\{1,\cdots,k\}$

$$\lim_{t\to 0^+}\frac{1}{t}\left[\int_{E^\varepsilon(q_j)} v(x,t)\,dx - \int_{E^\varepsilon(q_s)} v(x,t)\,dx\right] = 0. \tag{3.13}$$

*Proof.* Since the angles of $\Omega$ at two distinct vertices $q_j$ and $q_s$ are equal to one another, by a translation and an orthogonal transformation we can superpose one angle on the other one. Thus, there exists an orthogonal matrix $T$ such that the function $w=w(x,t)$ defined by

$$w(x,t) = v(x,t) - v(q_s + T(x-q_j),t)$$

satisfies

$$w_t = \Delta w \quad \text{in} \quad \left(\Omega\cap\overline{B_{\delta_0}(q_j)}\right)\times(0,\infty), \tag{3.14}$$

$$w = 0 \quad \text{on} \quad \left(\partial\Omega\cap\overline{B_{\delta_0}(q_j)}\right)\times(0,\infty), \tag{3.15}$$

$$w = 0 \quad \text{on} \quad \left(\Omega\cap\overline{B_{\delta_0}(q_j)}\right)\times\{0\}. \tag{3.16}$$

Since $\overline{\Omega}\cap\partial B_{\delta_0}(q_j)$ does not contain any vertices of $\Omega$, it follows from (2.8), (2.9), and (2.10) that there exist two positive constants $G>0, H>0$ satisfying

$$|w(x,t)| \le Ge^{-\frac{H}{t}}\ \text{ for all }\ (x,t)\in\left(\overline{\Omega}\cap\partial B_{\delta_0}(q_j)\right)\times(0,\infty). \tag{3.17}$$

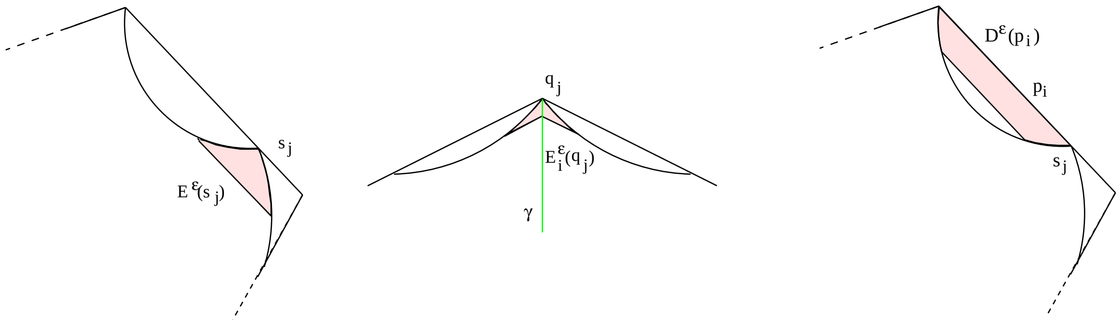



Figure 2: The sets $E^\varepsilon(s_j)$, $E^\varepsilon(q_j)$, and $D^\varepsilon(p_i)$.

Observe that

$$(\partial_t - \Delta)\left(Ge^{-\frac{H}{t}}\right) = GHt^{-2}e^{-\frac{H}{t}} > 0 \quad \text{for} \quad (x,t) \in \mathbb{R}^2 \times (0,\infty). \tag{3.18}$$

Therefore, in view of (3.14)-(3.18), by the comparison principle we obtain

$$|w(x,t)| \le Ge^{-\frac{H}{t}} \text{ for all } (x,t) \in (\Omega \cap B_{\delta_0}(q_j)) \times (0,\infty). \tag{3.19}$$

Since for $t > 0$

$$\begin{aligned}
&\frac{1}{t}\left|\int_{E^\varepsilon(q_j)} v(y,t)\ dy - \int_{E^\varepsilon(q_s)} v(y,t)\ dy\right| \\
&= \frac{1}{t}\left|\int_{E^\varepsilon(q_j)} v(y,t)\ dy - \int_{E^\varepsilon(q_j)} v(q_s + T(x-q_j),t)\ dx\right| \\
&\le \frac{1}{t}\int_{\Omega\cap B_{\delta_0}(q_j)} |w(x,t)|\ dx,
\end{aligned}$$

(3.19) implies (3.13). ☐

**Lemma 3.4** *If $\varepsilon > 0$ is sufficiently small, then there exist a positive sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to 0$ as $n \to \infty$ and a number $\lambda \in [4\cot 2\alpha, 8\cot 2\alpha]$ such that for any $j \in \{1,\cdots,k\}$*

$$\lim_{n\to\infty} \frac{1}{t_n}\int_{E^\varepsilon(q_j)} v(x,t_n)\ dx = \lambda. \tag{3.20}$$

*Proof.* It is clear that (3.7) guarantees that there exist a positive sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to 0$ as $n \to \infty$ and a number $\lambda \in [4\cot 2\alpha, 8\cot 2\alpha]$ such that (3.20) holds for $j = 1$. Therefore it follows from Lemma 3.3 that (3.20) holds for any $j \in \{1,\cdots,k\}$. ☐

**Lemma 3.5** *Let*

$$\rho = \sqrt{(R^*)^2 - R^2} > 0 \tag{3.21}$$

*and, for $\varepsilon > 0$ and $1 \le i \le m$ set*

$$D^\varepsilon(p_i) = \{\ x \in D_i\ :\ 0 < (x-p_i)\cdot\nu_i < \varepsilon\ \},$$

*where $\nu_i$ is the interior unit normal vector to the side of $\Omega$ containing $p_i$ (see Fig. 2).*

*Then, if $\varepsilon$ is sufficiently small, we have that for $1 \le i \le m$*

$$\lim_{t\to 0^+} \frac{1}{t}\int_{D^\varepsilon(p_i)} (x-x^*)\ v(x,t)\ dx = 4\rho\nu_i = -\frac{4\rho}{R}\ p_i. \tag{3.22}$$

*Proof.* We shall consider three cases: (a) the set $\partial D_i \cap \{q_1, \dots, q_k\}$ is empty; (b) the set $\partial D_i \cap \{q_1, \dots, q_k\}$ has exactly one point; (c) the set $\partial D_i \cap \{q_1, \dots, q_k\}$ has exactly two points. The treatment of case (c) is completely similar to that of case (b), thus, its proof will not be provided.

(a) Since $\overline{D_i}$ does not contain any vertex of $\Omega$, (2.8), (2.9), and (2.10) imply that there exist two positive constants $A_i$ and $B_i$ such that

$$\left| v(x,t) - f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right)\right| \le A_i\, e^{-\frac{B_i}{t}} \quad \text{for all } \ x \in D_i, t>0. \tag{3.23}$$

Let $e_i$ be a unit vector orthogonal to $\nu_i$. If $\varepsilon$ is sufficiently small, we can parametrize $D^\varepsilon(p_i)$ as

$$D^\varepsilon(p_i) = \{x = p_i + z_1\,\nu_i + z_2\,e_i : 0 < z_1 < \varepsilon,\ \varphi_-(z_1) < z_2 < \varphi_+(z_1)\}, \tag{3.24}$$

where now $\varphi_-(0) = -\rho,\ \varphi_+(0) = \rho$, and $\varphi_-'(0) = \cot\alpha,\ \varphi_+'(0) = -\cot\alpha$. Note that $x^*$, the reflection of $x \in D^\varepsilon(p_i)$, is given by

$$x^* = p_i - z_1\,\nu_i + z_2\,e_i.$$

We compute:

$$\begin{aligned}
&\frac{1}{t}\int_{D^\varepsilon(p_i)} (x-x^*)\ f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right) dx = \\
&\frac{1}{t}\int_0^\varepsilon \left[2z_1\nu_i f\left(t^{-\frac{1}{2}}z_1\right)\int_{\varphi_-(z_1)}^{\varphi_+(z_1)} dz_2\right]\, dz_1 = \\
&2\nu_i \int_0^{t^{-\frac{1}{2}}\varepsilon} [\varphi_+(t^{\frac{1}{2}}\xi) - \varphi_-(t^{\frac{1}{2}}\xi)]\ \xi f(\xi)\ d\xi,
\end{aligned}$$

and hence by Lebesgue's dominated convergence theorem we get

$$\lim_{t\to 0^+}\frac{1}{t}\int_{D^\varepsilon(p_i)} (x-x^*)\ f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right) dx = 4\rho\nu_i\ \int_0^\infty \xi f(\xi)\ d\xi.$$

With the aid of (3.23) and (2.6), we obtain (3.22).

(b) As in case (a), we consider the parametrization $x = p_i + z_1\,\nu_i + z_2\,e_i$ of a point in the set $D^\varepsilon(p_i)$ given in (3.24); additionally, we will assume that $p_i - \rho e_i$ is the point of $\partial D_i \cap \{q_1, \dots, q_k\}$.

Take a small number $\delta \in (0, \varphi_-(\varepsilon) + \rho)$ and set

$$\begin{aligned}
D_+^\varepsilon(p_i) &= \{x : 0 < z_1 < \varepsilon,\ \max(\varphi_-(z_1), \delta-\rho) < z_2 < \varphi_+(z_1)\},\\
D_-^\varepsilon(p_i) &= \{x : 0 < z_1 < \varepsilon,\ \min(\varphi_-(z_1), \delta-\rho) < z_2 < \delta-\rho\}.
\end{aligned} \tag{3.25}$$

Then $D^\varepsilon(p_i) = D_+^\varepsilon(p_i) \cup D_-^\varepsilon(p_i)$.

Since $\overline{D_+^\varepsilon(p_i)}$ does not contain any vertex of $\Omega$, from (2.8), (2.9) and (2.10) it follows that for some positive constants $A_i^+$ and $B_i^+$

$$\left| v(x,t) - f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right)\right| \le A_i^+ e^{-\frac{B_i^+}{t}} \ \text{for all } \ x \in D_+^\varepsilon(p_i),\ t>0. \tag{3.26}$$

Since the point $p_i - \rho\ e_i$ is a vertex of $\Omega$, from (2.9) and (2.10) we have that for some positive constants $A_i^-$ and $B_i^-$

$$\begin{aligned}
&0 < v(x,t) \le 2\ f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right) + A_i^- e^{-\frac{B_i^-}{t}}\\
&\qquad \text{for all } \ x \in D_-^\varepsilon(p_i),\ t>0.
\end{aligned} \tag{3.27}$$

We now compute:

$$\frac{1}{t}\int_{D_+^\varepsilon(p_i)}(x-x^*)\ f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right)dx =$$

$$\frac{2\nu_i}{t}\int_0^\varepsilon z_1 f\left(t^{-\frac{1}{2}}z_1\right)[\varphi_+(z_1)-\max(\varphi_-(z_1),\delta-\rho)]\ dz_1 =$$

$$2\nu_i\int_0^{t^{-\frac{1}{2}}\varepsilon}[\varphi_+(t^{\frac{1}{2}}\xi)-\max(\varphi_-(t^{\frac{1}{2}}\xi),\delta-\rho)]\ \xi f(\xi)\ d\xi,$$

and hence, by Lebesgue's dominated convergence theorem and (2.6), we get

$$\lim_{t\to 0^+}\frac{1}{t}\int_{D_+^\varepsilon(p_i)}(x-x^*)\ f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right)dx = 2(2\rho-\delta)\nu_i.$$

As before, we conclude that

$$\lim_{t\to 0^+}\frac{1}{t}\int_{D_+^\varepsilon(p_i)}(x-x^*)\ v(x,t)\ dx = 2(2\rho-\delta)\nu_i. \tag{3.28}$$

On the other hand, we have

$$\frac{1}{t}\int_{D_-^\varepsilon(p_i)}|x-x^*|\ f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right)dx =$$

$$\frac{2}{t}\int_0^\varepsilon z_1 f\left(t^{-\frac{1}{2}}z_1\right)[\delta-\rho-\min(\varphi_-(z_1),\delta-\rho)]\ dz_1 =$$

$$2\int_0^{t^{-\frac{1}{2}}\varepsilon}[\delta-\rho-\min(\varphi_-(t^{\frac{1}{2}}\xi),\delta-\rho)]\ \xi f(\xi)\ d\xi,$$

and hence by Lebesgue's dominated convergence theorem we get

$$\lim_{t\to 0^+}\frac{1}{t}\int_{D_-^\varepsilon(p_i)}|x-x^*|\ f\left(t^{-\frac{1}{2}}(x-p_i)\cdot\nu_i\right)dx = 2\delta.$$

Therefore (3.27) implies that

$$\limsup_{t\to 0^+}\left|\frac{1}{t}\int_{D_-^\varepsilon(p_i)}(x-x^*)\ v(x,t)\ dx\right| \le 4\delta$$

and thus, by (3.28), we get:

$$\limsup_{t\to 0^+}\left|\frac{1}{t}\int_{D^\varepsilon(p_i)}(x-x^*)\ v(x,t)\ dx - 4\rho\nu_i\right| \le 6\delta.$$

Since $\delta>0$ is chosen arbitrarily small, we again obtain (3.22). ∎

We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** In view of Lemma 2.1, it suffices to consider the integrals in (2.14) over the unions of the sets $E^\varepsilon(s_j)\cup E^\varepsilon(q_j)$ and $D^\varepsilon(p_i)$, respectively, for $\varepsilon>0$

sufficiently small. Lemma 2.1 thus guarantees that

$$\lim_{n\to\infty} \frac{1}{t_n}\left[\int_E xv(x,t_n)\ dx + \int_D (x-x^*)\ v(x,t_n)\ dx\right] =$$

$$\lim_{n\to\infty} \frac{1}{t_n}\sum_{j=1}^{\ell}\int_{E^\varepsilon(s_j)} xv(x,t_n)\ dx + \lim_{n\to\infty} \frac{1}{t_n}\sum_{j=1}^{k}\int_{E^\varepsilon(q_j)} xv(x,t_n)\ dx$$

$$+ \lim_{n\to\infty} \frac{1}{t_n}\sum_{i=1}^{m}\int_{D^\varepsilon(p_i)} (x-x^*)v(x,t_n)\ dx.$$

Lemmas 3.1, 3.2, 3.4, and 3.5 yield that

$$\lim_{n\to\infty} \frac{1}{t_n}\left[\int_E xv(x,t_n)\ dx + \int_D (x-x^*)\ v(x,t_n)\ dx\right]$$

$$= 2\cot\alpha \sum_{j=1}^{\ell} s_j + \lambda \sum_{j=1}^{k} q_j - \frac{4\rho}{R}\sum_{i=1}^{m} p_i.$$

Therefore (2.13) implies

$$2\cot\alpha \sum_{j=1}^{\ell} s_j + \lambda \sum_{j=1}^{k} q_j - \frac{4\rho}{R}\sum_{i=1}^{m} p_i = 0$$

and, by using (1.8) and (3.2), we get

$$(\lambda - 4\cot\alpha)\sum_{j=1}^{k} q_j = 0.$$

Therefore, since $\lambda \in [4\cot 2\alpha, 8\cot 2\alpha]$, we obtain (1.9). This completes the proof of Theorem 1.4. □

# 4 The proof of Theorem 1.3

Let $C_p = \partial B_R(0)$ and $C_q = \partial B_{R^*}(0)$ be the circles containing the points $p_1, \dots, p_m$ and $q_1, \dots, q_k$ respectively. As already observed, since $\partial\Omega$ is circumscribed to $C_p$, all the angles of $\Omega$ at the vertices $q_1, \dots, q_k$ are equal to each other. Also, notice that (1.9) directly implies that $k \geq 2$.

(1) We distinguish four cases (see Fig. 3). (i) Let $k = 2$; then $q_1$ and $q_2$ are opposite. Label by $p_1, p_2, p_3$ and $p_4$ the points in $\partial\Omega \cap C_p$ lying on the sides of $\Omega$ issuing from $q_1$ and $q_2$. They must be the vertices of a rectangle centered at 0; hence $\sum_{i=1}^{4} p_i = 0$ and, by (1.8), $p_5 = 0$ — a contradiction.

(ii) If $k = 3$, $q_1, q_2$ and $q_3$ are the vertices of an equilateral triangle, that we call $\mathcal{T}$; $\Omega$ and $\mathcal{T}$ have at least one side in common. Then $C_p$ must be the inscribed circle of $\mathcal{T}$ and any side of $\Omega$ issuing from any vertex of $\Omega$ lying outside $C_q$ cannot intersect $C_p$, since it must lie outside $\mathcal{T}$ — a contradiction.

(iii) Let $k = 4$. Since (1.9) holds, the $q_j$'s must be pairwise opposite and also be the vertices of a rectangle, for they all lie on $C_q$. Such rectangle and $\Omega$ must have at least three sides in common (tangent to $C_p$); this fact implies that the $q_j$'s are the vertices of a square. Hence, two sides of $\Omega$ issuing from the vertex of $\Omega$ lying outside $C_q$ cannot intersect $C_p$ — a contradiction.

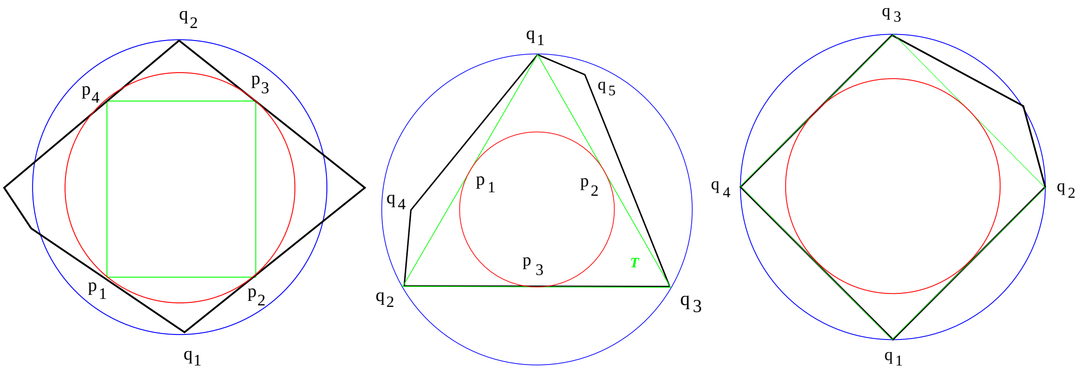



Figure 3: Proof of Theorem 1.3, item (1), cases (i), (ii) and (iii).

(iv) If $k = 5$, all the vertices of $\Omega$ lie on $C_q$ and hence all the angles of $\Omega$ must be equal to each other. The fact that $\partial\Omega$ is circumscribed to $C_p$ also implies that all the sides of $\Omega$ have equal length, that is $\Omega$ must be regular.

(2) We distinguish five cases (see Fig. 4). (i) If $k = 2$, then $q_1$ and $q_2$ are opposite. As in the proof of (2), $k = 2$, we let $p_1, p_2, p_3$ and $p_4$ be the points in $\partial\Omega \cap C_p$ lying on the sides of $\Omega$ issuing from $q_1$ and $q_2$. We have that $\sum_{i=1}^{4} p_i = 0$ and, by (1.8), it follows that $p_5 + p_6 = 0$. Therefore, all the points $p_i$'s are pairwise opposite and so are vertices of $\Omega$; hence, $\Omega$ is centrally symmetric. In other words, $\Omega$ is invariant under a rotation of an angle $\pi$.

(ii) If $k = 3$, $q_1, q_2$ and $q_3$ are the vertices of an equilateral triangle, that we call $\mathcal{T}$. If $\Omega$ and $\mathcal{T}$ have a side in common, then we get a contradiction, by the same argument used in the proof of (1), $k = 3$. If $\Omega$ and $\mathcal{T}$ have no side in common, then also the vertices of $\Omega$ lying outside $C_q$ must be the vertices of an equilateral triangle. In fact, since $\partial\Omega$ is circumscribed to $C_p$, such vertices lie on the three half-lines through the origin and the points $q_1 + q_2$, $q_2 + q_3$, and $q_3 + q_1$, respectively, and have the same distance from the origin. Therefore, $\Omega$ is invariant under a rotation of an angle $2\pi/3$.

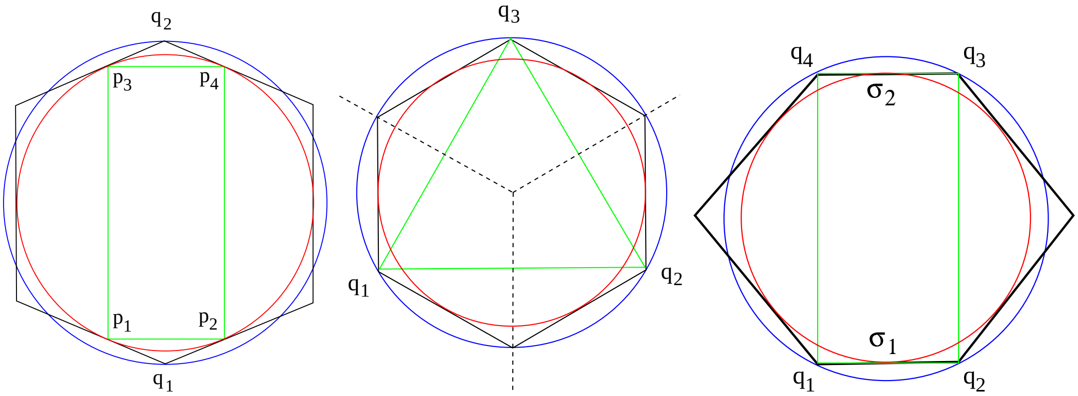



Figure 4: Proof of Theorem 1.3, item (2), cases (i), (ii) and (iii).

(iii) Let $k = 4$. Since (1.9) holds, the $q_j$'s must be pairwise opposite and also be the vertices of a rectangle $\mathcal{R}$, for they all lie on $C_q$. $\Omega$ and $\mathcal{R}$ have at least one side in common: let such a side be $\sigma_1$; $\sigma_1$ must be a shorter side of $\mathcal{R}$, since otherwise $C_p$ would be contained in $\mathcal{R}$ and hence at least one side of $\Omega$ would not intersect $C_p$. Thus, the side

$\sigma_2$ of $\mathcal{R}$ opposite to $\sigma_1$ must also be a side of $\Omega$ and the midpoints $p_1$ and $p_2$ of $\sigma_1$ and $\sigma_2$ are such that $p_1 + p_2 = 0$. By (1.8) we have that $\sum_{i=3}^{6} p_i = 0$. Therefore, the $p_i$'s are pairwise opposite and, as in the case $k = 2$, $\Omega$ is invariant under a rotation of an angle $\pi$.

(iv) The case $k = 5$ cannot occur. We can assume that the segments joining $q_1$ to $q_2$, $q_2$ to $q_3$, $q_3$ to $q_4$, and $q_4$ to $q_5$ are sides of $\Omega$. Since the angles of $\Omega$ at the points $q_j$'s are all equal to each other, we can suppose that $q_j = R^*(\cos(j-1)\theta, \sin(j-1)\theta),\ j = 1, \dots, 5$ for some positive angle $\theta$. Then (1.9) implies that $\theta = 2\pi/5$, that is the $q_j$'s are the vertices of a regular pentagon that contains $C_p$. Therefore, the sides of $\Omega$ issuing from the vertex of $\Omega$ outside $C_q$ cannot intersect $C_p$ because they lie outside the pentagon — a contradiction.

(v) If $k = 6$, all the vertices of $\Omega$ lie on $C_q$ and hence all the angles of $\Omega$ must be equal to each other. The fact that $\partial\Omega$ is circumscribed to $C_p$ also implies that all the sides of $\Omega$ have equal length, that is $\Omega$ must be regular.

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