Interaction between degenerate diffusion and shape of domain *

R. Magnanini†and S. Sakaguchi‡

September 14, 2005

Abstract

We consider the flow of a gas into a bounded tank Ω with smooth boundary $\partial\Omega$. Initially Ω is empty and at all times the density of the gas is kept constant on $\partial\Omega$. Choose a number $R > 0$ sufficiently small to have that, for any point x in Ω having distance R from $\partial\Omega$, the closed ball B with radius R centered at x intersects $\partial\Omega$ only at one point.

We show that if the gas content of such balls B is constant at each given time, then the tank Ω must be a ball. In order to prove this, we derive an asymptotic estimate for gas content for short times. Similar estimates are also derived in the case of the evolution p-Laplace equation for $p \geq 2$.

Key words. porous medium equation, gas content, initial behavior, symmetry of domains, evolution p-Laplacian, heat equation.

AMS subject classifications. Primary 35K55, 35K60; Secondary 35B40.

1 Introduction

We consider the flow of a gas into a bounded porous tank; the tank is initially empty and, at all times, the gas density is kept constant on the tank walls. This physical situation can be modeled as an initial-boundary value problem for a degenerate parabolic equation.

---

*This research was partially supported by Grants-in-Aid for Scientific Research (B) (♯ 12440042) and (B) (♯ 15340047) of Japan Society for the Promotion of Science, and by a PRIN Grant of the Italian MURST.

†Dipartimento di Matematica U. Dini, Università di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy. (magnanin@math.unifi.it).

‡Department of Mathematics, Faculty of Science, Ehime University, 2-5 Bunkyo-cho, Matsuyama-shi, Ehime 790-8577 Japan. (sakaguch@dpc.ehime-u.ac.jp).

We assume that our tank is represented by a bounded domain Ω in $\mathbb{R}^N$, $N \geq 2$, with smooth (say C^2) boundary $\partial\Omega$; the normalized density of gas at a point $x \in \Omega$ and time $t > 0$ is denoted by a function $u = u(x,t)$ satisfying the problem:

$$u_t = \Delta\phi(u) \quad \text{in} \quad \Omega \times (0,\infty), \tag{1.1}$$

$$u = 1 \quad \text{on} \quad \partial\Omega \times (0,\infty), \tag{1.2}$$

$$u = 0 \quad \text{on} \quad \Omega \times \{0\}, \tag{1.3}$$

where $\phi : [0,\infty) \to [0,\infty)$ is such that

$$\phi(0) = \phi'(0) = 0, \quad \phi \in C^1([0,\infty)) \cap C^2((0,\infty)), \tag{1.4}$$

$$\phi'(s) > 0 \ \text{ for } s > 0, \text{ and} \tag{1.5}$$

$$\int_0^1 \frac{\phi'(s)}{s}\, ds < \infty. \tag{1.6}$$

Existence and uniqueness of a bounded weak solution and the comparison principle are derived in [H] and [DK], with the aid of the regularity result of Sacks [S], together with the basic theory of quasilinear parabolic equations in [LSU]. It is known that condition (1.6) holds if and only if the equation $u_t = \Delta\phi(u)$ has the property of finite speed of propagation of disturbances from rest (see [P],[G]).

The purpose of this paper is to investigate how the shape of the tank influences the short time diffusion of gas from the tank walls. As an application of this investigation, we will prove a new symmetry result for the problem (1.1)-(1.3).

As a reference example of the situation considered here, the reader should keep in mind the case of the porous medium equation, in which $\phi(u) = u^m$ and $m > 1$ is a parameter. The property of finite speed of propagation of disturbances from rest implies that for any point $x \in \Omega$ there exists a time $T = T(x) > 0$ such that $u(x,t) = 0$ for all $t \in [0, T(x)]$ and $u(x,t) > 0$ for all $t > T(x)$.

When $\phi(u) = u^m$ with $m > 1$ and $\partial\Omega$ is of class C^4, the dependence of T on x has been estimated by C. Cortázar, M. Del Pino, and M. Elgueta (see Theorem 1.1 in [CDE]) in terms of the distance of x from $\partial\Omega$, that from now on we will denote by $d(x)$, and the mean curvature of $\partial\Omega$; in fact, they prove the estimate:

$$T(x) = d^2\{T_0 - (N-1)H(y(x))T_1 d + o(d)\}, \quad d = d(x),\ x \in \Omega_{\delta_0}. \tag{1.7}$$

Here, T_0 and T_1 are positive constants depending only on m;

$$\Omega_{\delta_0} = \{x \in \Omega \ : \ d(x) < \delta_0\}; \tag{1.8}$$

$\delta_0 > 0$ is chosen so small that

$$d \in C^2(\overline{\Omega_{\delta_0}}); \tag{1.9}$$

for every $x \in \Omega_{\delta_0}$ there exixts

$$\text{a unique } y = y(x) \in \partial\Omega \text{ such that } d(x) = |x - y|; \tag{1.10}$$

$$\max_{1 \le j \le N-1} \kappa_j(y) < \frac{1}{\delta_0} \quad \text{for any } y \in \partial\Omega. \tag{1.11}$$

Here and in the sequel, $\kappa_1(y), \cdots, \kappa_{N-1}(y)$ denote the principal curvatures of $\partial\Omega$ at $y \in \partial\Omega$ with respect to the interior normal direction to Ω, while

$$H(y) = \frac{1}{N-1} \sum_{j=1}^{N-1} \kappa_j(y)$$

is the mean curvature of $\partial\Omega$ at $y \in \partial\Omega$. (See [GT], Section 14.6, pp. 354–357.)

A straightforward conclusion that can be drawn from (1.7) is the following symmetry result.

Theorem 1.1 *Suppose that there exists a number $\delta > 0$ such that, for any pair of points $x_1, x_2 \in \Omega_\delta$, $d(x_1) = d(x_2)$ implies $T(x_1) = T(x_2)$.*

Then Ω must be a ball.

This result says that, if the gas flow reaches at the same time T points at equal distance from the tank's walls, then the tank has spherical shape. In fact, Theorem 1.1 is an easy consequence of V.I. Aleksandrov's *Soap Bubble Theorem* (see [Alek] p. 412, [R]), since its assumption implies that H must be constant on $\partial\Omega$.

In this paper we prove the symmetry result summarized in Theorem 1.2 below. Notice preliminarily that, if R is a positive number such that $R < \delta_0$, then for every point x in the parallel set $\Gamma_R = \{z \in \Omega : d(z) = R\}$ to $\partial\Omega$, the closure of the ball $B(x, R) = \{z \in \mathbb{R}^N : |z - x| < R\}$ intersects $\partial\Omega$ only at the point $y(x)$ defined in (1.10).

Theorem 1.2 *Suppose that, for every fixed time $t \in (0, 1)$ and any $x \in \Gamma_R$, the gas content of $B(x, R)$,*

$$\int_{B(x,R)} u(z, t)\, dz, \tag{1.12}$$

does not depend on x.

Then Ω must be a ball.

This result is based on the following asymptotic estimate.

Theorem 1.3 *For all $x \in \Gamma_R$, we have*

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}} \int\limits_{B(x,R)} u(z,t)\; dz = c(\phi, N) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y(x)) \right] \right\}^{-\frac{1}{2}}, \qquad (1.13)$$

where $y(x) \in \partial\Omega$ is the point defined in (1.10) *and $c(\phi, N)$ is a positive constant depending only on ϕ and N.*

Remark 1.4 When $N = 3$ in Theorem 1.3, the constant $c(\phi, N)$ is given by $c(\phi, 3) = 2\pi\phi(1)$ (see Section 4). Hence, if $\phi(1) = 1$, then $c(\phi, 3)$ just equals $c(3)$ $(= 2\pi)$, where $c(3)$ is the constant for the heat equation in Theorem 4.2 in Section 4.

The assumption of Theorem 1.2 implies that the right-hand side of (1.13) must be constant on $\partial\Omega$ and hence, again, we can use V.I. Aleksandrov's theorem to infer the symmetry of Ω.

In [MS], we proved an estimate similar to (1.13) for solutions of the resolvent equation $\Delta u - su = 0$, when the parameter $s \to +\infty$. In the present paper, besides deriving (1.13) for solutions of (1.1)-(1.3), with quite general assumptions on ϕ, we also propose a different and simpler proof. As in [MS], (1.13) is a consequence of the presence of a boundary layer for the solution of (1.1)-(1.3) when $t \to 0^+$.

Technically, (1.13) is obtained by working on integrals of the form

$$\int\limits_{B(x,R)} F(t^{-\frac{1}{2}} d(z))\; dz,$$

which bound the gas content (1.12) from above and below; F is determined in such a way that $F(t^{-\frac{1}{2}} d(z))$ is either a supersolution or a subsolution of (1.1)-(1.3) and its construction is simpler than the one worked out in [CDE].

The paper is organized as follows. In Section 2, we prove the asymptotic formula (2.1) on which (1.13) is based; the supersolutions and subsolutions for (1.1)-(1.3) are constructed in Section 3.

In Section 3, we also consider the heat and the evolution p-Laplace equation with $p > 2$,

$$u_t = \text{ div } (|\nabla u|^{p-2} \nabla u),$$

and find useful super and subsolutions for it. While the latter is another example of degenerate diffusion equation that has the property of finite speed of propagation of disturbances from rest, for the former such a speed is infinite and some extra work is needed. The technical details needed for all these results are proved in Section 5.

Finally, in Section 4, we prove our symmetry result, Theorem 1.3, and asymptotic estimates similar to (1.13) for the heat and the evolution p-Laplace equation with $p > 2$ (see Theorems 4.1 and 4.2).

2 Asymptotics

The following asymptotic formula is crucial to establish the initial behavior of u.

Lemma 2.1 *Let $\partial\Omega$ be of class C^2, $y \in \partial\Omega$, and $B(x,R)$ be an open ball centered at x and with radius $R > 0$ such that $\overline{B(x,R)} \cap (\mathbb{R}^N \setminus \Omega) = \{y\}$. Suppose that $\kappa_j(y) < \frac{1}{R}$ for $j = 1, \dots, N-1$, where $\kappa_1(y), \cdots, \kappa_{N-1}(y)$ denote the principal curvatures of $\partial\Omega$ at $y \in \partial\Omega$ with respect to the interior normal direction to Ω. Then we have:*

$$\lim_{s\to 0^+} s^{-\frac{N-1}{2}} \mathcal{H}^{N-1}(\Gamma_s \cap B(x,R)) = 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y) \right] \right\}^{-\frac{1}{2}}, \quad (2.1)$$

where $\mathcal{H}^{N-1}$ is the standard $(N-1)$-dimensional Hausdorff measure, and ω_{N-1} is the volume of unit ball in $\mathbb{R}^{N-1}$.

Proof. By suitably translating and rotating the coordinate axes, we can suppose that $y = 0$, the tangent space $T_y(\partial\Omega)$ to $\partial\Omega$ at y coincides with the hyperplane $z_N = 0$, the exterior unit normal vector $\nu(y)$ to $\partial\Omega$ at y points in the negative z_N direction, and $x = (0, \cdots, 0, R)$. By a further rotation around the z_N-axis, we can also choose the coordinates $z_1, \dots, z_{N-1}$ in such a way the function d satisfies the formulas

$$d(z) = z_N - \frac{1}{2} \sum_{j=1}^{N-1} \kappa_j(y)\ z_j^2 + o(|z|^2), \quad (2.2)$$

$$\frac{\partial d(z)}{\partial z_N} = 1 + o(|z|). \quad (2.3)$$

Notice that, with this choice of coordinates, the ball $B(x,R)$ is represented by the inequality $|z'|^2 + (z_N - R)^2 < R^2$, where $z' = (z_1, \dots, z_{N-1})$. Then, near the origin, $\partial B(x,R)$ is represented by

$$z_N = R - \sqrt{R^2 - |z'|^2} = \frac{1}{2R}\ |z'|^2 + O(|z'|^3). \quad (2.4)$$

Combining (2.2) with (2.4) yields

$$d(z) = \frac{1}{2} \sum_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y) \right) z_j^2 + o(|z'|^2) \ \text{ for } z \in B(0,R) \cap \partial B(x,R). \quad (2.5)$$

Since $\overline{B(x,R)} \cap (\mathbb{R}^N \setminus \Omega) = \{0\}$, for any $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that

$$\Gamma_s \cap B(x,R) \subset B(0,\varepsilon) \quad \text{if } \ 0 < s < s_\varepsilon. \quad (2.6)$$

Thus, because of (2.3), if $\varepsilon > 0$ is sufficiently small and $0 < s < s_\varepsilon$, $\Gamma_s \cap B(x, R)$ is represented by the graph of a smooth function $z_N = \psi(z')$. Differentiating $d(z', \psi(z')) = s$ with respect to z_j yields

$$d_{x_N} \nabla_{z'} \psi + \nabla_{z'} d = 0,$$

which together with $|\nabla d| = 1$ implies that

$$\sqrt{1 + |\nabla_{z'} \psi|^2} = 1/d_{x_N} \tag{2.7}$$

Projecting $\Gamma_s \cap B(x, R)$ orthogonally on the plane $z_N = 0$ yields a domain $A_s \subset \mathbb{R}^{N-1}$. Let $\eta > 0$ be sufficiently small. In view of (2.5) and (2.6), there exists $\varepsilon_0 > 0$ such that, for any $0 < s < s_{\varepsilon_0}$, we have

$$E_s^+ \subset A_s \subset E_s^-, \tag{2.8}$$

where $E_s^\pm$ are two ellipsoids defined by

$$E_s^\pm = \{z' \in \mathbb{R}^{N-1} \ : \ \frac{1}{2} \sum_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y) \pm \eta \right) z_j^2 < s\}. \tag{2.9}$$

Also, combining (2.3) with (2.7) yields that

$$1 \leq \sqrt{1 + |\nabla_{z'} \psi|^2} < 1 + \eta, \tag{2.10}$$

for each $0 < s < s_{\varepsilon_0}$. Hence, it follows from (2.8) and (2.10) that

$$\int_{E_s^+} 1 \ dz' \leq \mathcal{H}^{N-1}(\Gamma_s \cap B(x, R)) \leq \int_{E_s^-} (1 + \eta) \ dz', \tag{2.11}$$

for any $0 < s < s_{\varepsilon_0}$, since

$$\mathcal{H}^{N-1}(\Gamma_s \cap B(x, R)) = \int_{A_s} \sqrt{1 + |\nabla_{z'} \psi|^2} \ dz'.$$

Thus, from (2.11) we see that

$$2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y) + \eta \right] \right\}^{-\frac{1}{2}} \leq s^{-\frac{N-1}{2}} \mathcal{H}^{N-1}(\Gamma_s \cap B(x, R))$$
$$\leq 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y) - \eta \right] \right\}^{-\frac{1}{2}},$$

for any $0 < s < s_{\varepsilon_0}$.

Since $\eta > 0$ is arbitrarily small, we conclude that (2.1) holds. $\square$

3 Super and Subsolutions

In this section, we shall construct super and subsolutions for problem (1.1)-(1.3).

The same techniques can be adapted to derive super and subsolutions for the $p-$Laplace equation with $p > 2$; this will be done with Theorem 3.2. Similarly to (1.1), also this equation has the property of finite speed of propagation of disturbances from rest.

Some extra work is needed instead when such a speed is infinite as in the heat equation. In this case, we need to take care of the exponentially vanishing behavior of the solution inside the domain. We shall do this in Lemma 3.5.

Our construction of super and subsolutions for problem (1.1)-(1.3) is much simpler than that in [CDE], and hence we can deal with more general equations. By a result of Atkinson and Peletier (see Theorem 1 in [AtP]), there exist a number $a > 0$ and a classical solution $f = f(\xi)$ of the following boundary value problem:

$$(\phi'(f)f')' + \frac{1}{2}\xi f' = 0 \text{ in } [0,a), \tag{3.1}$$

$$f(0) = 1,\ f(\xi) \to 0,\ \phi'(f)f'(\xi) \to 0 \text{ as } \xi \to a, \tag{3.2}$$

$$\text{and } f(\xi) > 0,\ f'(\xi) < 0 \text{ in } [0,a). \tag{3.3}$$

Obviously in (3.2), ξ tends to a from below. We define a function $F = F(\xi)$ $(\xi \geq 0)$ by

$$F(\xi) = \begin{cases} f(\xi) & \text{if } 0 \leq \xi < a, \\ 0 & \text{if } \xi \geq a. \end{cases} \tag{3.4}$$

Let $0 < \varepsilon < \frac{a}{8}$. With the aid of the function $F = F(\xi)$, by the same argument used in [AtP], Theorem 1, we can also find two positive numbers $a_\pm$ and two classical solutions $f_\pm = f_\pm(\xi)$ of the following boundary value problem (see the end of Section 5 for the proof):

$$\left(\phi'(f_\pm)f'_\pm\right)' + \frac{1}{2}(\xi \mp 2\varepsilon)f'_\pm = 0 \text{ in } [0,a_\pm), \tag{3.5}$$

$$f_\pm(0) = 1,\ f_\pm(\xi) \to 0,\ \phi'(f_\pm)f'_\pm(\xi) \to 0 \text{ as } \xi \to a_\pm, \tag{3.6}$$

$$\text{and } f_\pm(\xi) > 0,\ f'_\pm(\xi) < 0 \text{ in } [0,a_\pm). \tag{3.7}$$

Obviously in (3.6), ξ tends to $a_\pm$ from below. We define two functions $F_\pm = F_\pm(\xi)$ $(\xi \geq 0)$ by

$$F_\pm(\xi) = \begin{cases} f_\pm(\xi) & \text{if } 0 \leq \xi < a_\pm, \\ 0 & \text{if } \xi \geq a_\pm. \end{cases} \tag{3.8}$$

Now, we set

$$w_\pm(x,t) = F_\pm\left(t^{-\frac{1}{2}}d(x)\right) \text{ for } (x,t) \in \Omega \times (0,\infty), \tag{3.9}$$

and, by Lemma 5.1, show that they are super and subsolutions for (1.1)-(1.3).

Theorem 3.1 *Let u be the solution of problem* (1.1)-(1.3)*. For each $\varepsilon \in (0, \frac{a}{8})$, there exists $t_\varepsilon > 0$ such that*

$$w_- \leq u \leq w_+ \quad \text{in } \ \Omega \times (0, t_\varepsilon],$$

where $w_\pm$ are defined by (3.9).

Proof. By Lemma 5.1 we have

$$0 < a_- \leq a \leq a_+ < \frac{5}{4}\ a \ \text{ for any } \ \varepsilon \in (0, \frac{a}{8}); \tag{3.10}$$

then we set

$$\tau = \left(\frac{\delta_0}{4a}\right)^2, \tag{3.11}$$

where $\delta_0 > 0$ is determined in (1.8)-(1.11).

Hence, if $0 < \varepsilon < \frac{a}{8}$ and $0 < t \leq \tau$, then

$$\{\ x \in \Omega \ : \ w_-(x,t) > 0\ \} \subset \{\ x \in \Omega \ : \ w_+(x,t) > 0\ \} \subset \Omega_{\frac{1}{2}\delta_0} \subset \Omega_{\delta_0}.$$

With the aid of this and (1.9), a straightforward computation gives

$$\begin{aligned}(w_\pm)_t - \Delta\phi(w_\pm) &= -\frac{1}{t} f'_\pm \left(\pm\varepsilon + t^{\frac{1}{2}}\phi'(f_\pm)\Delta d\right)\\ \text{in } \ &\bigcup_{0<t\leq\tau} \{\ x \in \Omega \ : \ w_\pm(x,t) > 0\ \} \times \{t\}.\end{aligned} \tag{3.12}$$

For each $\varepsilon \in (0, \frac{a}{8})$, set

$$t_\varepsilon = \min\left\{\tau, \left(\frac{\varepsilon}{2\Phi D}\right)^2\right\}, \tag{3.13}$$

where $\Phi = \max\limits_{0\leq s\leq 1} \phi'(s)$ and $D = \max\limits_{x\in\overline{\Omega_{\delta_0}}} |\Delta d(x)|$. Thus, in view of the definition of $f_\pm$, by using (3.10) and (3.12), we conclude that

$$\begin{aligned}(\pm 1)\Big\{(w_\pm)_t - \Delta\phi(w_\pm)\Big\} &> 0\\ \text{in } \ \bigcup_{0<t\leq t_\varepsilon} \{\ x \in \Omega \ : \ w_\pm(x,t) &> 0\ \} \times \{t\}.\end{aligned} \tag{3.14}$$

This implies that w_+ and w_- are weak super and subsolutions for problem (1.1)-(1.3) in $\Omega \times (0, t_\varepsilon]$, and hence the comparison principle completes the proof. □

We now proceed to derive similar comparison results for the evolution $p-$Laplace equation with $p > 2$, that is, we want to consider the unique weak solution u of the initial-boundary value problem:

$$u_t = \text{div}\,(|\nabla u|^{p-2}\nabla u) \quad \text{in } \Omega \times (0,\infty), \tag{3.15}$$

$$u = 1 \quad \text{on } \partial\Omega \times (0,\infty), \tag{3.16}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}. \tag{3.17}$$

See [DiB] for existence and uniqueness results for this problem.

For $\lambda \geq \xi \geq 0$, define $\varphi = \varphi_\lambda(\xi)$ by

$$\varphi_\lambda(\xi) = 1 - \left[\frac{p-2}{2p(p-1)}\right]^{\frac{1}{p-2}} \int_0^\xi (\lambda^2 - \eta^2)^{\frac{1}{p-2}} d\eta. \tag{3.18}$$

A positive number $\xi_0 > 0$ is determined by the equation $\varphi_{\xi_0}(\xi_0) = 0$. Then $\varphi = \varphi_{\xi_0}(\xi)$ satisfies the following:

$$(p-1)|\varphi'|^{p-2}\varphi'' + \frac{1}{p}\varphi'\xi = 0 \quad \text{in } [0,\xi_0), \tag{3.19}$$

$$\varphi(0) = 1,\ \varphi'(\xi_0) = \varphi(\xi_0) = 0, \tag{3.20}$$

$$\text{and } \varphi' < 0 \quad \text{in } [0,\xi_0). \tag{3.21}$$

We define the function $F = F(\xi)$ $(\xi \geq 0)$ by

$$F(\xi) = \begin{cases} \varphi(\xi) & \text{if } 0 \leq \xi \leq \xi_0, \\ 0 & \text{if } \xi > \xi_0. \end{cases} \tag{3.22}$$

For each $\varepsilon \in (0, \frac{\xi_0}{2p})$, when $\lambda \geq \xi \geq 0$ and $\lambda \geq \max\{\pm 2p\varepsilon, 0\}$, define $\psi = \psi_{\pm,\lambda}(\xi)$ by

$$\psi_{\pm,\lambda}(\xi) = 1 - \left(\frac{p-2}{2p(p-1)}\right)^{\frac{1}{p-2}} \int_0^\xi \{(\lambda - \eta)(\lambda + \eta \mp 2p\varepsilon)\}^{\frac{1}{p-2}}\, d\eta. \tag{3.23}$$

Two positive numbers $\xi_\pm > 0$ are determined by the equations $\psi_{\pm,\xi_\pm}(\xi_\pm) = 0$. Compared with $\varphi_{\xi_0}(\xi_0) = 0$, we see that $\xi_+ > \xi_0$ $(> 2p\varepsilon)$. This guarantees the existence of $\xi_+ \geq 2p\varepsilon$. Then $\psi = \psi_{\pm,\xi_\pm}(\xi)$ satisfy the following problems:

$$(p-1)|\psi'|^{p-2}\psi'' + \frac{1}{p}(\xi \mp p\varepsilon)\psi' = 0 \quad \text{in } [0,\xi_\pm), \tag{3.24}$$

$$\psi(0) = 1,\ \psi'(\xi_\pm) = \psi(\xi_\pm) = 0, \tag{3.25}$$

$$\text{and } \psi' < 0 \quad \text{in } [0,\xi_\pm), \tag{3.26}$$

respectively.

We define two functions $F_\pm = F_\pm(\xi)$ $(\xi \geq 0)$ by

$$F_\pm(\xi) = \begin{cases} \psi_{\pm,\xi_\pm}(\xi) & \text{if } 0 \leq \xi \leq \xi_\pm, \\ 0 & \text{if } \xi > \xi_\pm. \end{cases} \tag{3.27}$$

By setting

$$w_\pm(x,t) = F_\pm\left(t^{-\frac{1}{p}} d(x)\right) \quad \text{for } (x,t) \in \Omega \times (0,\infty), \tag{3.28}$$

with the help of Lemma 5.2, we obtain the following result.

Theorem 3.2 *Let u be the solution of problem* (3.15)-(3.17). *For each $\varepsilon \in (0, \frac{\xi_0}{2p})$, there exists $t_\varepsilon > 0$ satisfying*

$$w_- \leq u \leq w_+ \quad \text{in } \Omega \times (0, t_\varepsilon],$$

where $w_\pm$ are defined by (3.28).

Proof. By Lemma 5.2 we have

$$0 < \xi_- < \xi_0 < \xi_+ < \frac{3}{2}\xi_0. \tag{3.29}$$

Then we set

$$\tau = \left(\frac{\delta_0}{4\xi_0}\right)^p, \tag{3.30}$$

where $\delta_0 > 0$ is determined in (1.8)-(1.11). Hence, if $0 < \varepsilon < \frac{\xi_0}{2p}$ and $0 < t \leq \tau$, then

$$\{\, x \in \Omega \;:\; w_-(x,t) > 0 \,\} \subset \{\, x \in \Omega \;:\; w_+(x,t) > 0 \,\} \subset \Omega_{\frac{1}{2}\delta_0} \subset \Omega_{\delta_0}.$$

With the aid of this and (1.9), a straightforward computation gives

$$\begin{aligned} (w_\pm)_t - \text{ div } (|\nabla w_\pm|^{p-2}\nabla w_\pm) &= -\frac{1}{t}\psi'_{\pm,\xi_\pm}\left(\pm\varepsilon + t^{\frac{1}{p}}|\psi'_{\pm,\xi_\pm}|^{p-2}\Delta d\right) \\ \text{in } \bigcup_{0<t\leq\tau} &\{\, x \in \Omega \;:\; w_\pm(x,t) > 0 \,\} \times \{t\}. \end{aligned} \tag{3.31}$$

For each $\varepsilon \in (0, \frac{\xi_0}{2p})$, set

$$t_\varepsilon = \min\left\{\tau, \left(\frac{p(p-1)\varepsilon}{4\xi_0^2(p-2)D}\right)^p\right\}, \tag{3.32}$$

where $D = \max_{x \in \overline{\Omega_{\delta_0}}} |\Delta d(x)|$. Thus, in view of the definition (3.23) of $\psi_{\pm,\xi_\pm}$, by using (3.29) and (3.31) we conclude that

$$\begin{aligned} (\pm 1)\Big\{(w_\pm)_t - \text{ div } (|\nabla w_\pm|^{p-2}\nabla w_\pm)\Big\} &> 0 \\ \text{in } \bigcup_{0<t\leq t_\varepsilon} \{\, x \in \Omega \;:\; w_\pm(x,t) > 0 \,\} &\times \{t\}. \end{aligned} \tag{3.33}$$

This implies that w_+ and w_- are weak super and subsolutions for problem (3.15)-(3.17) in $\Omega \times (0, t_\varepsilon]$, and hence the comparison principle completes the proof. □

Finally, we shall construct super and subsolutions for the initial-boundary value problem for the heat equation:

$$u_t = \Delta u \qquad \text{in} \;\; \Omega \times (0, \infty), \tag{3.34}$$

$$u = 1 \qquad \text{on} \;\; \partial\Omega \times (0, \infty), \tag{3.35}$$

$$u = 0 \qquad \text{on} \;\; \Omega \times \{0\}. \tag{3.36}$$

Since heat equation has the property of infinite speed of propagation of disturbances from rest, we need to take care of the inside of Ω. With the aid of the linearity of heat equation, we can overcome this difficulty. Define $F = F(\xi)$ $(\xi \geq 0)$ by

$$F(\xi) = \frac{1}{\sqrt{\pi}} \int_\xi^\infty e^{-\frac{1}{4}s^2} ds \quad \left(= 1 - \frac{1}{\sqrt{\pi}} \int_0^\xi e^{-\frac{1}{4}s^2} ds \right). \tag{3.37}$$

Then F satisfies the following properties:

$$F'' + \frac{1}{2}\xi F' = 0 \;\; \text{in} \;\; [0, \infty), \tag{3.38}$$

$$F(0) = 1, \;\; F(\xi) \to 0 \;\; \text{as} \;\; \xi \to \infty, \tag{3.39}$$

$$\text{and} \;\; F' < 0 \;\; \text{in} \;\; [0, \infty). \tag{3.40}$$

Also, for each $\varepsilon \in (0, 1)$, we define two functions $F_\pm = F_\pm(\xi)$ $(\xi \geq 0)$ by

$$F_\pm(\xi) = \frac{1}{C_\pm} \int_\xi^\infty e^{-\frac{1}{4}(s \mp 2\varepsilon)^2} ds \quad \left(= 1 - \frac{1}{C_\pm} \int_0^\xi e^{-\frac{1}{4}(s \mp 2\varepsilon)^2} ds \right), \tag{3.41}$$

where $C_\pm = \int_{\mp 2\varepsilon}^\infty e^{-\frac{1}{4}s^2} ds$. Then $F_\pm$ satisfy the following properties

$$F_\pm'' + \frac{1}{2}(\xi \mp 2\varepsilon) F_\pm' = 0 \;\; \text{in} \;\; [0, \infty), \tag{3.42}$$

$$F_\pm(0) = 1, \;\; F_\pm(\xi) \to 0 \;\; \text{as} \;\; \xi \to \infty, \tag{3.43}$$

$$\text{and} \;\; F_\pm' < 0 \;\; \text{in} \;\; [0, \infty), \tag{3.44}$$

respectively.

By setting

$$v_\pm(x, t) = F_\pm\left(t^{-\frac{1}{2}} d(x)\right) \quad \text{for} \;\; (x, t) \in \Omega \times (0, \infty), \tag{3.45}$$

we obtain the following result.

Lemma 3.3 *For each* $\varepsilon \in (0,1)$, *there exists* $t_{1,\varepsilon} > 0$ *satisfying*

$$(\pm 1)\left\{(v_\pm)_t - \Delta v_\pm\right\} > 0 \quad \textit{in} \;\; \Omega_{\delta_0} \times (0, t_{1,\varepsilon}].$$

Proof. With the aid of (1.9), a straightforward computation gives

$$(v_\pm)_t - \Delta v_\pm = -\frac{1}{t}\left(\pm\varepsilon + \sqrt{t}\Delta d\right)\; F'_\pm \quad \text{in} \;\; \Omega_{\delta_0} \times (0,\infty).$$

Then, for each $\varepsilon \in (0,1)$, by setting

$$t_{1,\varepsilon} = \left(\frac{\varepsilon}{2D}\right)^2,$$

where $D = \max_{x \in \overline{\Omega_{\delta_0}}} |\Delta d(x)|$, we complete the proof. □

Let u be the solution of problem (3.34)-(3.36). A result of Varadhan [V] shows that

$$-4t \log u(x,t) \to d(x)^2 \;\; \text{as} \;\; t \to 0^+ \;\; \text{uniformly on} \;\; \overline{\Omega}. \tag{3.46}$$

Then, in view of this and the definition (3.45) of $v_\pm$, we have

Lemma 3.4 *There exist three positive constants* t_0, E_1 *and* E_2 *satisfying*

$$\max\{|v_+|,\; |v_-|,\; |u|\} \le E_1 e^{-\frac{E_2}{t}} \quad \textit{in} \;\; \overline{\Omega \setminus \Omega_{\delta_0}} \times (0, t_0],$$

where u *is the solution of problem* (3.34)-(3.36).

Proof. If we choose $t_0 \in (0, \left(\frac{\delta_0}{4}\right)^2]$, then by (3.45) we can show the desired inequalities for $v_\pm$. As for u, by (3.46) we can choose $t_0 > 0$ such that

$$\left|4t \log u(x,t) + d(x)^2\right| < \frac{1}{2}\delta_0^2 \;\; \text{for} \;\; (x,t) \in \overline{\Omega} \times (0, t_0],$$

and hence

$$u(x,t) < e^{-\frac{d(x)^2 - \frac{1}{2}\delta_0^2}{4t}} \;\; \text{for} \;\; (x,t) \in \overline{\Omega} \times (0, t_0].$$

Since $d(x) \ge \delta_0$ for $x \in \overline{\Omega \setminus \Omega_{\delta_0}}$, we get the desired inequality for u. □

By setting

$$w_\pm(x,t) = v_\pm(x,t) \pm E_1 e^{-\frac{E_2}{t}} \;\; \text{for} \;\; (x,t) \in \Omega \times (0,\infty), \tag{3.47}$$

we have the following result.

Theorem 3.5 *Let* u *be the solution of problem* (3.34)-(3.36). *For each* $\varepsilon \in (0,1)$, *there exists* $t_\varepsilon > 0$ *satisfying*

$$w_- \le u \le w_+ \quad \textit{in} \;\; \Omega \times (0, t_\varepsilon],$$

where $w_\pm$ *are defined by* (3.47).

Proof. For each $\varepsilon \in (0,1)$, we set

$$t_\varepsilon = \min\{t_{1,\varepsilon}, t_0\}.$$

Since $v_+,\ v_-,$ and u are all nonnegative, Lemma 3.4 implies that

$$w_- \le u \le w_+ \quad \text{in } \overline{\Omega \setminus \Omega_{\delta_0}} \times (0, t_\varepsilon]. \tag{3.48}$$

Observe that

$$w_- \le u \le w_+ \qquad \text{on } \partial\Omega \times (0, t_\varepsilon], \tag{3.49}$$

$$w_- = u = w_+ = 0 \qquad \text{on } \Omega \times \{0\}. \tag{3.50}$$

Therefore, with the aid of the comparison principle, in view of Lemma 3.3, (3.48), (3.49), and (3.50), we complete the proof. □

4 Symmetry results

We begin with the proof of Theorem 1.3 together with Remark .

Proof. Let $x_0 \in \Gamma_R$ and put $y_0 = y(x_0) \in \partial\Omega$. Take $\varepsilon \in (0, \frac{a}{8})$. By Theorem 3.1 we get for any $t \in (0, t_\varepsilon)$

$$\int_{B(x_0,R)} w_-(x,t)\ dx \le \int_{B(x_0,R)} u(x,t)\ dx \le \int_{B(x_0,R)} w_+(x,t)\ dx. \tag{4.1}$$

Integrating on the level surfaces of d by the coarea formula gives:

$$\int_{B(x_0,R)} w_\pm(x,t)\ dx = \int_0^{2R} F_\pm\left(t^{-\frac{1}{2}}s\right) \mathcal{H}^{N-1}(\Gamma_s \cap B(x_0,R))\ ds,$$

where $\Gamma_s = \{x \in \Omega : d(x) = s\}$. Setting $s = t^{\frac{1}{2}}\xi$ yields that

$$t^{-\frac{N+1}{4}} \int_{B(x_0,R)} w_\pm(x,t)\ dx$$

$$= \int_0^{2Rt^{-\frac{1}{2}}} F_\pm(\xi)\xi^{\frac{N-1}{2}} \cdot\ (t^{\frac{1}{2}}\xi)^{-\frac{N-1}{2}} \mathcal{H}^{N-1}(\Gamma_{t^{\frac{1}{2}}\xi} \cap B(x_0,R))\ d\xi.$$

Then, it follows from Lemma 2.1 and Lebesgue's dominated convergence theorem that

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}} \int_{B(x_0,R)} w_\pm(x,t)\ dx$$

$$= \int_0^{a_\pm} f_\pm(\xi)\xi^{\frac{N-1}{2}} d\xi \cdot 2^{\frac{N-1}{2}} \omega_{N-1} \left\{\prod_{j=1}^{N-1}\left[\frac{1}{R} - \kappa_j(y_0)\right]\right\}^{-\frac{1}{2}}.$$

Since $\varepsilon > 0$ is arbitrarily small, combining this with (4.1) and Lemma 5.1 yields

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}} \int\limits_{B(x_0,R)} u(x,t)\, dx$$

$$= \int_0^a f(\xi)\xi^{\frac{N-1}{2}} d\xi \cdot 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y_0) \right] \right\}^{-\frac{1}{2}}.$$

This shows that (1.13) holds with $c(\phi, N) = 2^{\frac{N-1}{2}} \omega_{N-1} \int_0^a f(\xi)\, \xi^{\frac{N-1}{2}} d\xi$.

In particular, when $N = 3$, $c(\phi, 3) = 2\pi \int_0^a f(\xi)\xi\, d\xi$. Multiplying equation (3.1) by ξ and integrating by parts yield that

$$-\int_0^a (\phi(f))'\, d\xi - \int_0^a f(\xi)\xi\, d\xi = 0,$$

where (3.2) was used. This shows that $c(\phi, 3) = 2\pi\phi(1)$. $\Box$

The same arguments used in this proof, with slight modifications, lead to the proofs of the two theorems below.

Theorem 4.1 *Let u be the solution of problem* (3.15)-(3.17). *For all $x \in \Gamma_R$, we have*

$$\lim_{t\to 0^+} t^{-\frac{N+1}{2p}} \int\limits_{B(x,R)} u(z,t)\, dz = c(p,N) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y(x)) \right] \right\}^{-\frac{1}{2}}, \qquad (4.2)$$

where $\kappa_1, \dots, \kappa_{N-1}$ denote the principal curvatures of $\partial\Omega$ with respect to the interior normal direction to $\partial\Omega$, $y(x) \in \partial\Omega$ is the point defined in (1.10), *and*

$$c(p,N) = \frac{2^{\frac{N-1}{2}} \omega_{N-1}}{N+1} \left[\frac{p-2}{2p(p-1)} \right]^{\frac{1}{p-2}} B\left(\frac{N+3}{4}, \frac{p-1}{p-2} \right) \xi_0^{\frac{N+3}{2} + \frac{2}{p-2}}.$$

Here B is Euler's beta function and ξ_0 is that defined by $\varphi_{\xi_0}(\xi_0) = 0$ where $\varphi_\lambda(\xi)$ is given by (3.18).

Theorem 4.2 *Let u be the solution of problem* (3.34)-(3.36). *For all $x \in \Gamma_R$, we have*

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}} \int\limits_{B(x,R)} u(z,t)\, dz = c(N) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y(x)) \right] \right\}^{-\frac{1}{2}}, \qquad (4.3)$$

where $\kappa_1, \dots, \kappa_{N-1}$ denote the principal curvatures of $\partial\Omega$ with respect to the interior normal direction to $\partial\Omega$, $y(x) \in \partial\Omega$ is the point defined in (1.10), *and*

$$c(N) = \frac{2^{N+1}}{\sqrt{\pi}(N+1)} \Gamma\left(\frac{N+3}{4} \right) \omega_{N-1}.$$

Here Γ is Euler's gamma function.

5 Technical lemmas

A comparison argument yields the following result.

Lemma 5.1 *Let F and $F_\pm$ be defined by* (3.1)-(3.4) *and* (3.5)-(3.8), *respectively.*

Then the following assertions hold:

(i) *for each $\varepsilon \in (0, \frac{a}{8})$,*

$$0 < a-2\varepsilon \le a_- \le a \le a_+ \le a+2\varepsilon \ \text{ and } \ 0 \le F_- \le F \le F_+ \ \text{ in } \ [0,\infty);$$

(ii) *as $\varepsilon \downarrow 0$, $F_\pm \to F$ uniformly on $[0,\infty)$.*

Proof. Let $\varepsilon \in (0, \frac{a}{8})$. Write

$$v_\pm = \phi(F_\pm) \quad \text{and} \quad v = \phi(F). \tag{5.1}$$

Then, for each $\xi \in [0, a)$ or for each $\xi \in [0, a_\pm)$, integrating equation (3.1) and (3.5) over the interval $[0, \xi]$ yields that

$$v'(\xi) - v'(0) + \frac{1}{2}\int_0^\xi \eta F'(\eta)\, d\eta = 0,$$
$$v'_\pm(\xi) - v'_\pm(0) + \frac{1}{2}\int_0^\xi (\eta \mp 2\varepsilon) F'_\pm(\eta)\, d\eta = 0.$$

Furthermore, since $F(0) = F_\pm(0) = 1$, we get by an integration by parts

$$v'(\xi) = v'(0) - \frac{1}{2}\xi F(\xi) + \frac{1}{2}\int_0^\xi F(\eta)\, d\eta. \tag{5.2}$$

$$v'_\pm(\xi) = v'_\pm(0) - \frac{1}{2}\big[(\xi \mp 2\varepsilon)F_\pm(\xi) \pm 2\varepsilon\big] + \frac{1}{2}\int_0^\xi F_\pm(\eta)\, d\eta. \tag{5.3}$$

Note that (3.2) and (3.6) implies that (5.2) and (5.3) hold also for $\xi = a$ and for $\xi = a_\pm$, respectively. Hence by the definition of F and $F_\pm$ we conclude that both (5.2) and (5.3) hold for any $\xi \in [0,\infty)$.

Let us show that

$$F_- \le F \le F_+ \ \text{ in } \ [0,\infty). \tag{5.4}$$

Note that this implies that

$$a_- \le a \le a_+. \tag{5.5}$$

Suppose that $F \le F_+$ does not hold. Since $F(0) = F_+(0) = 1$ and $F(\xi) = F_+(\xi) = 0$ for $\xi \ge \max\{a, a_+\}$, there exists an open finite interval (α, β) in $[0,\infty)$ satisfying

$$F > F_+ \ \text{ on } \ (\alpha,\beta) \ \text{ and } \ F = F_+ \text{ at } \ \{\alpha,\beta\}. \tag{5.6}$$

In particular, this yields that

$$v'(\alpha) \geq v'_+(\alpha) \text{ and } v'(\beta) \leq v'_+(\beta). \tag{5.7}$$

By using (5.2) and (5.3), we have

$$v'(\beta) - v'(\alpha) = -\frac{1}{2}\big[\beta F(\beta) - \alpha F(\alpha)\big] + \frac{1}{2}\int_\alpha^\beta F(\eta)\, d\eta, \tag{5.8}$$

$$v'_+(\beta) - v'_+(\alpha) = -\frac{1}{2}\big[(\beta - 2\varepsilon)F_+(\beta) - (\alpha - 2\varepsilon)F_+(\alpha)\big] + \frac{1}{2}\int_\alpha^\beta F_+(\eta)\, d\eta.$$

Therefore, since $F = F_+$ at $\{\alpha, \beta\}$, we conclude that

$$v'(\beta) - v'_+(\beta) - \big[v'(\alpha) - v'_+(\alpha)\big] = -\varepsilon[F(\beta) - F(\alpha)] + \frac{1}{2}\int_\alpha^\beta [F(\eta) - F_+(\eta)]\, d\eta.$$

Since F is non-increasing, by (5.6) the right-hand side of this equality is positive, which contradicts (5.7). This shows that $F \leq F_+$ holds true.

By the same argument we can show that $F_- \leq F$ also holds true, and hence we complete the proof of (5.4).

We now prove (ii) and the first set of inequalities in (i) by using the two auxiliary functions $G_\pm = G_\pm(\xi)(\xi \geq 0)$ defined by

$$G_+(\xi) = \begin{cases} F(\xi - 2\varepsilon) & \text{if } \xi \geq 2\varepsilon, \\ 1 & \text{if } 0 \leq \xi \leq 2\varepsilon, \end{cases} \tag{5.9}$$

$$G_-(\xi) = F(\xi + 2\varepsilon) \text{ for } \xi \geq 0. \tag{5.10}$$

In fact, we will show that

$$G_- \leq F_- \text{ and } F_+ \leq G_+ \text{ in } [0, \infty), \tag{5.11}$$

since this, together with (5.4) and (5.5), clearly yields (ii) and

$$a - 2\varepsilon \leq a_- \leq a \leq a_+ \leq a + 2\varepsilon.$$

Set $V_\pm = \phi(G_\pm)$ and suppose that $F_+ \leq G_+$ does not hold. Since $G_+ = 1$ and $F_+ < 1$ on $(0, 2\varepsilon]$ and $F_+(\xi) = G_+(\xi) = 0$ for $\xi \geq \max\{a_+, a + 2\varepsilon\}$, there exists an open finite interval (α, β) in $[2\varepsilon, \infty)$ satisfying

$$F_+ > G_+ \text{ on } (\alpha, \beta) \text{ and } F_+ = G_+ \text{ at } \{\alpha, \beta\}. \tag{5.12}$$

Since (5.8) holds true for any finite interval (α, β) in $[0, \infty)$, we have that

$$V'_+(\beta) - V'_+(\alpha) = -\frac{1}{2}\big\{(\beta - 2\varepsilon)G_+(\beta) - (\alpha - 2\varepsilon)G_+(\alpha)\big\} + \frac{1}{2}\int_\alpha^\beta G_+(\eta)\, d\eta.$$

Hence, by using this instead of (5.8), by the same comparison argument as in the proof of $F \le F_+$, we obtain a contradiction and conclude that $F_+ \le G_+$ holds true.

By this same argument, inequality $G_- \le F_-$ easily follows .□

By similar comparison arguments, we can prove the following two lemmas, whose proofs are omitted.

Lemma 5.2 *Let F and $F_\pm$ be defined by* (3.19)-(3.22) *and* (3.24)-(3.27), *respectively.*

Then the following assertions hold:

(i) *for each $\varepsilon \in (0, \frac{\xi_0}{2p})$, we have that $0 < \xi_0 - p\,\varepsilon \le \xi_- < \xi_0 < \xi_+ \le \xi_0 + p\,\varepsilon$ and*

$$0 \le F_- \le F \le F_+ \quad in \quad [0, \infty);$$

(ii) *as $\varepsilon \downarrow 0$, $F_\pm \to F$ uniformly on $[0, \infty)$.*

Lemma 5.3 *Let F and $F_\pm$ be defined by* (3.37) *and* (3.41), *respectively.*

Then for each $\varepsilon \in (0, 1)$, $0 \le F_- \le F \le F_+$ in $[0, \infty)$, and as $\varepsilon \downarrow 0$, $F_\pm \to F$ uniformly on $[0, \infty)$.

Finally, we proceed to the proof of the existence of two positive numbers $a_\pm$ and two classical solutions $f_\pm = f_\pm(\xi)$ of problem (3.5)-(3.7).

Proof. First of all, since we are concerned with the solutions $f_\pm$ satisfying $0 \le f_\pm \le 1$, we may modify the function $\phi = \phi(s)$ for large $s > 1$ to have

$$\int_1^\infty \frac{\phi'(s)}{s}\, ds = \infty. \tag{5.13}$$

This corresponds to the condition A in [AtP], p. 370. The existence of a_- and f_- follows from the same argument used in [AtP], since ξ is replaced by $\xi + 2\varepsilon$ in equation (3.1) and $\xi + 2\varepsilon$ is nonnegative in $[0, \infty)$. Here, we consider only a_+ and f_+. Let $\varepsilon \in (0, \frac{a}{8})$ and let $\tilde{a} \in [\frac{3}{4}a, \frac{5}{4}a]$. Since $2\varepsilon < \tilde{a}$, by the same argument used in [AtP], Theorem 1, there exists a unique $b(\tilde{a}) > 0$ and a unique classical solution $f(\xi) = f(\xi; \tilde{a})$ of the following boundary value problem:

$$(\phi'(f)f')' + \frac{1}{2}(\xi - 2\varepsilon)f' = 0 \quad \text{in} \quad [0, \tilde{a}), \tag{5.14}$$

$$f(0) = b(\tilde{a}),\ f(\xi) \to 0,\ \phi'(f)f'(\xi) \to 0 \quad \text{as} \quad \xi \to \tilde{a}, \tag{5.15}$$

$$\text{and} \quad f(\xi) > 0,\ f'(\xi) < 0 \quad \text{in} \quad [0, \tilde{a}). \tag{5.16}$$

Obviously in (5.15), ξ tends to $\tilde{a}$ from below. Moreover, $b(\tilde{a})$ is monotonically increasing and continuous in $\tilde{a}$ on $[\frac{3}{4}a, \frac{5}{4}a]$. Therefore, in view of the intermediate value theorem, in order to prove the existence of a_+ and f_+, it is sufficient for us to show that $b(\frac{3}{4}a) < 1 < b(\frac{5}{4}a)$. For this purpose, we define the function $F(\xi; \tilde{a})$ $(\xi \geq 0)$ by

$$F(\xi; \tilde{a}) = \begin{cases} f(\xi; \tilde{a}) & \text{if } 0 \leq \xi < \tilde{a}, \\ 0 & \text{if } \xi \geq \tilde{a}. \end{cases} \tag{5.17}$$

Suppose that $b(\frac{3}{4}a) \geq 1$. Then, it follows from the same comparison argument as in the proof of Lemma 5.1 that $F(\xi) \leq F(\xi; \frac{3}{4}a)$ in $[0, \infty)$, where $F(\xi)$ is given by (3.4). This yields that $a \leq \frac{3}{4}a$, a contradiction. Similarly, suppose that $b(\frac{5}{4}a) \leq 1$. Then, by the same comparison argument, we have that $F(\xi; \frac{5}{4}a) \leq G_+(\xi)$ in $[0, \infty)$, where $G_+(\xi)$ is given by (5.9). This yields that $\frac{5}{4}a \leq a + 2\varepsilon (< \frac{5}{4}a)$, a contradiction. Thus we conclude that $b(\frac{3}{4}a) < 1 < b(\frac{5}{4}a)$. □

References

[Alek] A. D. Aleksandrov, Uniqueness theorems for surfaces in the large V, Vestnik Leningrad Univ. 13, no. 19 (1958), 5–8. (English translation: Amer. Math. Soc. Transl., Ser. 2, 21 (1962), 412–415.)

[AtP] F. V. Atkinson and L. A. Peletier, Similarity profiles of flows through porous media, Arch. Rational Mech. Anal. 42 (1971), 369–379.

[CDE] C. Cortázar, M. Del Pino, and M. Elgueta, On the short-time behavior of the free boundary of a porous medium equation, Duke Math. J. 87 (1997), 133–149.

[DK] B. E. J. Dahlberg and C. E. Kenig, Non-negative solutions of generalized porous medium equations, Rev. Mat. Iberoamericana 2 (1986), 267–305.

[DiB] E. DiBenedetto, Degenerate Parabolic Equations, Springer-Verlag, Berlin, Heidelberg, New York, 1993.

[GT] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, (Second Edition.), Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

[G] B. H. Gilding, The occurrence of interfaces in nonlinear diffusion-advection processes, Arch. Rational Mech. Anal. 100 (1988), 243–263.

[H] Zhongdan Huan, Some basic facts for a class of degenerate parabolic equations, Differential Integral Equations 5 (1992), 961–972.

[LSU] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Transl. Math. Monogr., Vol. 23, Amer, Math. Soc., Providence, 1968.

[MS] R. Magnanini and S. Sakaguchi, Matzoh ball soup: Heat conductors with a stationary isothermic surface, Ann. of Math. (2) 156 (2002), 931–946.

[P] L. A. Peletier, A necessary and sufficient condition for the existence of an interface in flows through porous media, Arch. Rational Mech. Anal. 56 (1974), 183–190.

[R] R. C. Reilly, Mean curvature, the Laplacian, and soap bubbles, Amer. Math. Monthly 89 (1982), 180–188, 197–198.

[S] P. Sacks, Continuity of solutions of a singular parabolic equation, Nonlinear Anal. 7 (1983), 387–409.

[V] S. R. S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, Comm. Pure Appl. Math. 20 (1967), 431–455.