Weighted finite element method of high degree of accuracy for Dirichlet problem with singularity

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Boundary value problems with strongly singular solutions (we say that u is strongly singular if $u \notin H^1$ or the Dirichlet integral of u is divergent) occur in the physics of plasmas and gas discharges, nuclear physics and other fields of physics. In [1] a singular boundary value problem was investigated, for which a generalized (weak) solution in the Sobolev space $H^1(\Omega)$ could not be defined, or did not have enough regularity. For example, it is well known that the generalized solution of the boundary value problem for an elliptic equation in a two-dimensional domain Ω with a boundary containing reentrant angles γ_i $(i = 1, 2, \dots, N)$ belongs to the class $H_2^{1+k-\epsilon}(\Omega)$. Here, $k = \min\limits_{i=1,\dots,N}\{k_i\}$, where $k_i = \dfrac{\pi}{\gamma_i}$ for a Dirichlet or Neumann problem and $k_i = \dfrac{\pi}{2\gamma_i}$ for a mixed boundary value problem, and ε is any positive number. In this case, according to the principle of coordinated estimates, the finite-element or finite-difference approximate solution converges to a generalized solution of the problem at an $O(h^k)$ rate in the norm of the space $H_2^1(\Omega)$.

For the boundary value problems with singularity we offered to define the solution as R_ν-generalized one. In [2–4] a finite element method without loss of accuracy for these boundary value problems was constructed and investigated. It was shown that the approximation to the R_ν-generalized solution converges with first and second order in norms of the Sobolev and Lebesque weighted spaces respectively.

In this paper we construct a scheme of the finite element method of high degree of accuracy for the boundary value problem with non-coordinated degeneration of input data and singularity of solution. The rate of convergence of an approximate solution of the proposed finite element method to the exact R_ν-generalized solution in the weight set $W^1_{2,\nu+\beta/2+2}(\Omega, \delta)$ is investigated, the estimation of finite element approximation $O(h^2)$ is established. The theoretical estimate is illustrated by numerical results obtained for a series of model problems.

References

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